

Advanced Political Economy

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Chapter 1

Introduction

1.1 Preamble

These lecture notes were taken during the summer semester (2026) at the Economics Department of Roma Tre University, of the *Advanced Political Economy* course, adapted from the slides (and class notes) of Professor Saverio M. Fratini. **Each section/chapter has its own notation unless specified otherwise.** All mathematical definitions and tools used in these lecture notes are recovered from Meyer (2023).

1.2 Ricardo's Theory of Extensive Rent

In a capitalist system of production, *rent* is paid for the use of natural resources; it arises from the existence of property rights. Three kinds of rent: (i) **absolute** rent represents the minimum amount of “something” that the owner of a certain natural resource has to receive to allow the use of the latter; (ii) **differential rent** is the consequence of scarcity, two or more methods are operated for the production of the same commodity; (iii) **rent paid by a monopoly price** occurs when the scarcity of natural resource, plus the particular conditions of production of a certain commodity, set a limit on its produced quantity.

Differential rent can be *extensive* or *intensive*. Here we analyze the notion of extensive

differential rent. Let $\mathcal{T} = \{1, 2, \dots, n\}$, with $n \in \mathbb{N}$ be the set of qualities of land we have

$$c_k \oplus 1T_k \rightarrow x \quad \forall k \in \mathcal{T}.$$

This last expression is the technique of associating, describing its fertility, i.e., the amount of corn c_k necessary to produce a given quantity of corn $x > c_k$ with strictly one acre of land $1T_k$. With the fertility of each land ordered like $c_1 < c_2 < \dots < c_n$. Let $C \in N$ be the demand for corn in the economy, and let T_k^* be the amount of total available land of type $k \in \mathcal{T}$. To supply demand:

$$C = x \left(\sum_{k=1}^n T_k \right).$$

Where $T_k \in [0, T_k^*]$ is the amount of acres effectively used of $k \in \mathcal{T}$ to meet demand. Let ρ_k be the rent associated to $k \in \mathcal{T}$. If $T_k < T_k^*$, then competition among landowners forces $\rho_k = 0$. The rate of profit of any k -th type of land is given by

$$r_k(\rho_k) = \frac{x - c_k - \rho_k}{c_k} \quad \text{with} \quad \frac{dr_k}{d\rho_k} < 0$$

Let land $n \in \mathcal{T}$ be the least fertile land, i.e., $x > c_n > c_k \forall k = 1, 2, \dots, n - 1$. And let $T_k = T_k^* \forall k = 1, 2, \dots, n - 1$, such that land n is the only one that doesn't command a rent ($T_n < T_n^*$): its rate of profit is not a function of ρ_n . Then, according to Ricardo's theory of extensive differential rent ρ_k^* must be such that $r_k(\rho_k^*) = r_n = r^* \forall k = 1, 2, \dots, n - 1$, hence

$$\rho_k^* = x \frac{c_n - c_k}{c_n} = (x - c_k) - r^* c_k$$

Differential rent per acre in land $k \in \{1, 2, \dots, n - 1\}$ is the output per acre x scaled by the relative fertility gap to the marginal land $\frac{c_n - c_k}{c_n}$. Or, we could also say that rent is the residual of the net product $x - c_k$ after the uniform profit rate is charged on that land's capital $r^* c_k$.

Chapter 2

The Notion of Surplus

2.1 Physical Real Cost and Surplus

In classical political economy, production and exchange are a circular process: commodities are produced by means of commodities. **The physical real cost** C of any commodity is formed by the commodities which are necessary for the production (e.g., workers' subsistence, raw materials, tools, and machines) of a given amount of output X of such commodity. **Surplus** S is the *net excess of production*, i.e., the difference between outputs and physical real costs: $S = X - C$. In a capitalist economy, S is divided into three parts such that $S = R + W^* + T$, where R are profits, W^* are wages above subsistence, and T are rents. Each type of income is associated with a specific social class: capitalists, workers, and landowners.

2.2 The Physiocrats

The notion of surplus was also adopted by the French physiocrats. According to their analysis: (i) output in the agricultural sector is the only one to include a surplus (in light of the *cooperation* of nature and humans); (ii) there are three social classes: proprietary class (landowners), productive class (farmers) and sterile class (artisans, merchants); (iii) the process of production and consumption is circular.

The physiocrats were unable to grasp the presence of a surplus in the manufacturing

sector because of the heterogeneity between outputs (diverse manufactured goods) and inputs (agricultural commodities); this is not the case in the agricultural sector, where outputs and inputs are physically homogeneous. In this approach, it must be said that surplus is equivalent to the rent earned by the proprietary class; this is a feature of a pre-capitalistic society.

2.3 Adam Smith

Smith indeed studied a capitalist system. The latter is characterized by: (i) commodities are produced to be sold on the market; (ii) commodities have a natural price which emerges from the repetition of its production; and (iii) national wealth Y corresponds to the sum of the natural price of all the commodities produced on a given year (that is a flow). In Smith's analysis, we have three social classes (types of income in which the national wealth is divided): workers (wages), capitalists (profits), and landlords (rents).

Since the national wealth corresponds to the natural value of the produced commodities, both consist of three parts: wages, profit, and rent, i.e., $Y = rK + wL + \rho T$. Each commodity has a **nominal** (in terms of money, affected by its purchasing power) and a **real** price (the exchange value of the commodity). Let $N = \{1, 2, \dots, n\}$ be the set of distinct commodities of an economy; the exchange value of i in terms of j is $\frac{p_i}{p_j}$, for $i \neq j$ and $i, j \in N$. Let w be the wage; then for any $i \in N$, we have that $\frac{p_i}{w}$ is the amount of labour that i "commands", i.e., the amount of labour that i can purchase in exchange. Each commodity has a **natural** and a **market price**; market prices gravitate around its natural price. In this analysis, gross output X consists of capital (replacement of the means of production) and surplus; wages are advances. Hence $X = (K + wL) + S$.

In his analysis, we can find two categories of labour: productive L_p and unproductive L_u . The former is such that the wages of its workers are capital invested, while for the latter, wages are paid by the surplus (profit and rent); this implies that $S = r(K + wL) + \rho T$. Then $\frac{L_p}{L_u} = f\left(\frac{K}{S}\right)$, i.e., the proportion between productive and unproductive workers depends on the capital-surplus ratio $\frac{K}{S}$. The national wealth of an economy depends on (i) the number of productive workers employed and (ii) the productivity of labour, i.e., division of labour.

Division of labour is understood as the specialization of each worker in a particular job, which increases dexterity, saves time from switching between jobs, and fosters innovation by specialized workers. In Smith's analysis, the distribution of national income between workers and capitalists depends on **class conflict**.

2.4 David Ricardo

2.4.1 Essay on Profits

For David Ricardo, surplus S is a *social magnitude*, defined as $S = X - K$, with X being the national product and K being the necessary means of production to produce X ; special assumptions are required to determine S in a single sector. In Ricardo's agricultural sector, X_a and $K_a = wL_a$ (L_a being the necessary amount of labour to produce X_a) consist of corn, with w being a certain amount of corn, i.e., workers' subsistence in the form of wages (corn) is advanced, and it is the only necessary means of production in the agricultural sector. Surplus in the agricultural sector is $S_a = X_a - K_a$, with the rate of profits being

$$r_a = \frac{S_a}{K_a} = \frac{X_a - K_a}{K_a} = \frac{X_a}{K_a} - 1,$$

which is independent of prices under the assumption that wages only consist of corn. Let $N = \{1, 2, \dots, n\}$ be the set of sectors existing in the economy. For a commodity $i \in N$, if $K_i = wL_i$ (its necessary means of production only consist of workers' subsistence needs,) then

$$r_i = \frac{p_i X_i - p_a K_i}{p_a K_i} = \frac{p_i}{p_a} \cdot \frac{X_i}{K_i} - 1.$$

Where p_i is the price of commodity i and p_a is the price of corn, such that $\frac{p_i}{p_a}$ is the relative price of i in terms of corn. Ricardo outlines the following mechanisms:

- (i) If $r_i > r_a \Rightarrow \uparrow X_i \wedge \downarrow X_a \Rightarrow \downarrow \frac{p_i}{p_a} \Rightarrow r_i \rightarrow r_a$
- (ii) If $r_i < r_a \Rightarrow \downarrow X_i \wedge \uparrow X_a \Rightarrow \uparrow \frac{p_i}{p_a} \Rightarrow r_i \rightarrow r_a$

Hence, Ricardo concludes that the rate of profit of the corn sector determines the natural rate of profit in all other sectors of the economy, i.e., the general rate of profit $r = r_a$ depends on the productivity of labour in the corn sector.

2.4.2 Principles of Political Economy

Removing the assumption that workers' wages consist only of corn implies that sectoral profit rates cannot be determined independently of commodities' relative prices. Profits are not independent of relative prices; however, relative prices are also not independent of the natural levels of the distribution variables. Ricardo circumvented this problem with the **Labour Theory of Value**.

If (i) $K = wL$ consists exclusively of wages paid in advance for one period, and (ii) the rate of profit is uniform across sectors, then commodities' relative prices are independent of distribution, and can be derived via the technical conditions of production. Let $N = \{1, 2, \dots, n\}$ be the set of commodities in an economy. Now, $\forall i \in N$ the technique of production is

$$l_i \rightarrow 1X_i.$$

This means that l_i is commodity i labour coefficient, the amount of labour required to produce one unit of X_i (also known as labour *embodied*). Then, for $i \neq j$ with $i, j \in N$, the relative price of commodity i in terms of j is

$$\frac{p_i}{p_j} = \frac{l_i w(1+r)}{l_j w(1+r)} = \frac{l_i}{l_j}.$$

This implies that relative prices equal the relative quantity of labour embodied. Assume that $\exists i \in N$ such that $l_i = 1$, let this commodity be $\mu \in N$. Note that $L_i = X_i l_i \forall i \in N$. Assume as *numéraire* commodity μ , then the the national product in terms of μ can be written in the following form:

$$X = \frac{1}{p_\mu} \sum_{i=1}^n p_i X_i = \sum_{i=1}^n \frac{p_i}{p_\mu} X_i = \sum_{i=1}^n \frac{l_i w(1+r)}{w(1+r)} X_i = \sum_{i=1}^n l_i X_i = \sum_{i=1}^n L_i = L.$$

Let w be the wage rate in terms of labour embodied, given that $K = wL$, the profit rate can be takes the following form:

$$r = \frac{S}{K} = \frac{X - K}{K} = \frac{L - wL}{wL} = \frac{1}{w} - 1 \quad \text{with} \quad \frac{dr}{dw} < 0.$$

There is an inverse relationship between the rate of profits and the wage rate in terms of labour embodied. In this analysis, w depends on: (i) the quantities of commodities that form

workers' subsistence; and (ii) the productivity of labour of such subsistence commodities. Hence, r will fall if the quantity of commodities given to workers increases and/or the productivity of labour decreases.

2.5 Karl Marx

Capital K consists of two parts $K = K_c + K_v$, with K_c and $K_v = wL$ being the constant (value of commodities employed as means of production) and variable (value of commodities given to workers, in advance) capital. The organic composition of capital is given by $\Omega = \frac{K_c}{K_v}$. The value of the annual gross product is $X = L + K_c$, being the sum of directly L and indirectly K_c embodied labour. Surplus value is

$$S = X - K = (L + K_c) - (K_c + K_v) = L - K_v = L - wL = L(1 - w).$$

Dividing both sides of the last expression by K_v we obtain the **rate of surplus value** (i.e., the rate of exploitation):

$$s = \frac{S}{K_v} = \frac{L(1 - w)}{wL} = \frac{(1 - w)}{w} = \frac{1}{w} - 1.$$

Marx critiques Ricardo's theory of profit, which is defined as the rate of exploitation of the agricultural sector (see Section 2.4.1), given that in the corn model $r_a = \frac{S_a}{K_a}$ has only variable capital in the denominator. Hence, Ricardo conflates r with s by neglecting the role of constant capital. For Marx, the general rate of profit depends on both s and Ω . We define this profit rate as:

$$r = \frac{S}{K} = \frac{S}{K_c + K_v} = \frac{s}{\Omega + 1}, \text{ with } \frac{dr}{ds} > 0 \text{ and } \frac{dr}{d\Omega} < 0.$$

This last expression outlines the **tendency of the rate of profits to fall**. Attempts by capitalists to increase labour productivity via an increase in Ω reduce the general rate of profits with the condition of s being constant. However, w also may change with an "ambiguous" sign because of a substitution of past for living labour.

Finally, for the transformation of values into **prices of production**. Let $N = \{1, 2, \dots, n\}$ be the set of commodities produced in the economy. For each $i \in N$ write $K_i = K_{c,i} + K_{v,i}$

and $\Omega_i = \frac{K_{c,i}}{K_{v,i}}$, and assume a *uniform rate of exploitation* $s = \frac{S_i}{K_{v,i}} \forall i \in N$, so that surplus value is produced in strict proportion to living labour:

$$S_i = sK_{v,i}.$$

The value of commodity i is then $X_i = K_{c,i} + L_i = K_i + S_i$, since living labour $L_i = K_{v,i} + S_i$ splits into the reproduction of the wage advance and the appropriated surplus. If commodities were exchanged at their values, the sectoral profit rate would be

$$r_i^X = \frac{S_i}{K_i} = \frac{s K_{v,i}}{K_{c,i} + K_{v,i}} = \frac{s}{\Omega_i + 1},$$

which is strictly decreasing in Ω_i : sectors of higher organic composition would have lower profit rates. However, under free competition, the profit rate tends towards the general profit rate, i.e., total surplus value over total capital. With $S = \sum_{i=1}^n S_i$ and $K = \sum_{i=1}^n K_i$:

$$r = \frac{S}{K} = \frac{\sum_{i=1}^n s K_{v,i}}{\sum_{i=1}^n K_i} = \frac{s}{\bar{\Omega} + 1}, \quad \bar{\Omega} \equiv \frac{\sum_{i=1}^n K_{c,i}}{\sum_{i=1}^n K_{v,i}} \equiv \frac{K_c}{K_v}.$$

Imposing this uniform rate, the **price of production** of commodity i is

$$p_i = K_i(1 + r) = K_i\left(1 + \frac{S}{K}\right).$$

How is it that this transformation of value X_i to prices of production p_i redistributes surplus value across sectors? Each sector produces surplus value in proportion to its living labour ($s K_{v,i}$) but appropriate profit in proportion to its total advanced capital ($r K_i$). The net transfer accruing to sector i is the gap between price and value,

$$\delta_i \equiv p_i - X_i = K_i(1 + r) - (K_i + S_i) = r K_i - s K_{v,i},$$

whose sign is governed entirely by the organic composition relative to the social average:

$$\delta_i > 0 \iff \frac{1}{\bar{\Omega} + 1} = \frac{r}{s} > \frac{K_{v,i}}{K_i} = \frac{1}{\Omega_i + 1} \iff \frac{1}{\bar{\Omega} + 1} > \frac{1}{\Omega_i + 1} \iff \Omega_i > \bar{\Omega}.$$

A sector is therefore a net recipient of surplus value if and only if its organic composition exceeds the social-average composition $\bar{\Omega}$, and a net donor otherwise: surplus value flows from sectors of low to sectors of high organic composition of capital. Two conservation

properties are held by construction and constitute Marx's defense of the procedure, noting that $S = rK$:

$$\sum_{i=1}^n \delta_i = rK - sK_v = S - S = 0 \implies \sum_{i=1}^n p_i = \sum_{i=1}^n X_i, \quad \sum_{i=1}^n (p_i - K_i) = \sum_{i=1}^n S_i.$$

That is, total price of production equals total value, and total profit equals total surplus value: the transformation only reshuffles a conserved labour magnitude. It must be stressed that in $p_i = K_i(1+r)$ the capital K_i is reckoned at *value*: inputs enter at $K_{c,i}, K_{v,i}$ rather than at their own transformed prices of production. **This is the single-step transformation: outputs are transformed while inputs are not.** Once one requires inputs to exchange at prices of production as well (since one sector's output is another's input), the two aggregate equalities above cannot in general both hold simultaneously (the Bortkiewicz problem). The *direction* of the transfer, $\Omega_i \gtrless \bar{\Omega}$, is nonetheless robust to this objection.

Chapter 3

The Capitalist Mode of Production

3.1 A stylized Presentation of the Capitalist System

Definition 1 (Capital). Capital K is the amount of value (purchasing power) invested at the beginning of the production process to finance the costs of production.

Assumption 1 (Assumptions). (\mathfrak{A}_1) Society is organized in classes: capitalists, workers, and landowners. (\mathfrak{A}_2) Capitalists are the ruling class. (\mathfrak{A}_3) The capitalist system is a market system (outputs are produced to be sold in the market, inputs are purchased in the market). (\mathfrak{A}_4) The employment of inputs precedes the production of outputs, i.e., costs and revenues of the same process are not simultaneous. (\mathfrak{A}_5) K is the amount of purchasing power *required* to advance the costs of a given production process. (\mathfrak{A}_6) Profit is the difference between revenues and costs of each process.

Definition 2 (Technique of Production). Let $M \in \mathbb{N}$ commodities used as outputs and inputs; $A \in \mathbb{N}$ labour services; and $B \in \mathbb{N}$ productive services of natural resources. With $\mathbf{K}_t \in \mathbb{R}_+^M$, $\mathbf{L}_t \in \mathbb{R}_+^A$, $\mathbf{N}_t \in \mathbb{R}_+^B$ being the vectors of capital goods, labour services, and productive services of natural resources, respectively. Let $\mathbf{Y} \in \mathbb{R}_+^M$ be the vector of outputs, then, the production process is:

$$\mathbf{K}_t \oplus \mathbf{L}_t \oplus \mathbf{N}_t \rightarrow \mathbf{Y}_{t+1}$$

Definition 3 (Costs, Revenues, Profits, Rate of Profits). Let $\mathbf{p}_t \in \mathbb{R}_+^M$, $\mathbf{w}_t \in \mathbb{R}_+^A$ and $\boldsymbol{\rho}_t \in \mathbb{R}_+^B$ be the vectors of commodity prices, wages and rent rates in period t . Total costs

$C_t \in \mathbb{R}_+$ are:

$$\mathbf{p}_t^T \mathbf{K}_t + \mathbf{w}_t^T \mathbf{L}_t + \boldsymbol{\rho}_t^T \mathbf{N}_t = C_t.$$

Revenues are given by

$$\mathbf{p}_{t+1}^T \mathbf{Y}_{t+1} = R_{t+1}.$$

Profits (surplus) are given by

$$\Pi_{t+1} = R_{t+1} - C_t$$

The rate of profits is the ratio

$$r_{t,t+1} = \frac{\Pi_{t+1}}{C_t} = \frac{R_{t+1}}{C_t} - 1.$$

3.2 The Capitalist Circuit

In a pre-capitalistic economy, the circulation of commodities is $C - M - C$, with M being a certain amount of purchasing power (money): a commodity is sold for money in order to purchase another commodity of equal value, so money is a mere intermediary. In a capitalistic economy, the circulation process is $M - C - M'$: money is advanced to finance the costs of production before a process can be started, and at the end of the process, with the sale of outputs, capital becomes purchasing power again. With M the value of the costs advanced by capital, C the commodities purchased or produced by means of capital, and M' the value of the revenues generated at the end of the process, profit is the increment

$$\Pi = M' - M, \quad M' = M + \Pi.$$

The defining feature of capitalist circulation is precisely $M' > M$: the self-valorization (self-expansion) of capital. By contrast, a circuit $M - C - M$ with no increment would be pointless, since it returns exactly the value advanced. Some considerations regarding capital: (i) capital is not an input (the inputs are capital goods, labour services, and the productive services of natural resources); (ii) capital is an amount of value, of the same kind as costs and revenues; and (iii) the rate of profit is not the price of capital — profit is a residual.

3.3 Real Wages

From the classical political economy point of view, income distribution depends on class conflict. The social net product is always divided among three income categories: wages, rents, and profits. The *surplus*, however, comprises rents, profits, and *only that portion of wages in excess of subsistence*: subsistence wages belong to necessary consumption, not to the surplus. Hence, if $w = \bar{w}$ (the subsistence wage), the surplus reduces to rents and profits; if $w > \bar{w}$, the surplus has three components. Workers' bargaining power depends on: (i) economic, or technical, elements (the level and composition of aggregate demand); (ii) institutional elements (trade unions, contract forms, frictional unemployment); and (iii) social elements (habits, culture, class consciousness).

Wage determination is not a market phenomenon, i.e., there is no monotonically decreasing relationship between wages and the level of employment. Reducing wages does not lead to higher employment rates; unemployment affects wage rates indirectly (by reducing workers' bargaining power). There is a minimum level of wages (subsistence) below which the nominal wage rate cannot fall. **Social subsistence** is the quantity of commodities that enables worker reproduction, and it depends on social and institutional elements.

3.4 Natural Prices and the General Rate of Profit

We know that the surplus of an economy is $S = R - C$, and that $r = \frac{S}{K}$. For Smith, each commodity has a *market* and a *natural price*. The former is the price at which that commodity is actually exchanged in the market; the latter is the system of prices that allows the same proportion between profit and capital in every sector, i.e., a uniform rate of profits. Natural prices are theoretical variables, while market prices are a phenomenon based on the quantity actually brought to a market and effective demand.

These two types of prices are connected via **competition**. Competition among buyers (sellers) drives the market price *above* (below) the natural price when the produced quantity of a commodity is below (above) effectual demand. Because of this discrepancy between market and natural prices, the rates at which labour, land, and capital are paid cannot be

uniform across sectors; this in turn induces competition within each social class, creating a tendency towards uniform remunerations. The return of the market price *to* the natural price is a logically distinct second movement — **gravitation** — driven by the reallocation of capital across sectors.

Let p_i, p_j be the natural prices of commodities i, j . For $i, j \in N$, at any given t , we have $\tilde{p}_i \neq p_i$ and $\tilde{p}_j \neq p_j$, i.e., market prices \tilde{p} are not equal to natural prices, which implies that $\tilde{r}_i \neq \tilde{r}_j$. Let $\tilde{r}_i > \tilde{r}_j$; gravitation then takes the form:

$$\tilde{r}_i > \tilde{r}_j \Rightarrow \uparrow X_i \wedge \downarrow X_j \Rightarrow \downarrow \frac{\tilde{p}_i}{\tilde{p}_j} \Rightarrow \downarrow \tilde{r}_i \wedge \uparrow \tilde{r}_j.$$

This process only arrives at a rest position at $\tilde{p}_i = p_i$, $\tilde{p}_j = p_j$ and $\tilde{r}_i = \tilde{r}_j = r$.

3.5 The Core of the Surplus Approach

The *givens* of the surplus approach are: (i) the total product of the economy and its composition; (ii) the technique (or techniques of production) and the necessary means of production (K) and labour (L); (iii) the real wage rate (w). With these data, the theory can determine the level of employment (indirectly), the rate of profit, and the relative prices of commodities.

From here onwards, costs comprise the means of production and (subsistence) wages only; rent is treated as a component of the surplus rather than as a cost, so r , S and K are redefined accordingly, with K now denoting the means of production. Assume an economy producing only one good by means of labour and the same good (as circulating capital); given the available technique and the *real subsistence wage* in terms of that commodity, the surplus identity is

$$Y - (K + wL) = S = \Pi + \rho T, \quad (3.1)$$

where S is the income available after reproduction to pay profits (Π) and rents (ρT). Profit is reckoned on the advanced capital according to the wage convention adopted:

$$\Pi = \begin{cases} rK & \text{(wages not advanced),} \\ r(K + wL) & \text{(wages advanced).} \end{cases} \quad (3.2)$$

If $\rho T = 0$, so that land is over-abundant (and hence free), the whole surplus is profit income, and the rate of profit is

$$r = \frac{S}{K} \quad (\text{wages not advanced}), \quad (3.3)$$

$$r = \frac{S}{K + wL} \quad (\text{wages advanced}). \quad (3.4)$$

There is, accordingly, an inverse relationship between profits and wages. In the wages-not-advanced case, solving $Y - K - wL = rK$ for w gives

$$w = \frac{Y - (1 + r)K}{L}, \quad (3.5)$$

while under advanced wages the same residual yields $w = \frac{Y - (1 + r)K}{(1 + r)L}$.

Returning to the givens: (i) the determination of the total product is where the surplus approach *departs* from neoclassical analysis — the level of social product is governed by accumulation and effective demand, Say's Law is accepted (earned income is entirely spent on consumption or investment), but this does *not* entail full employment of labour, whereas the marginalist system pins output by full employment of given factor endowments. (ii) What the two approaches genuinely *share* attaches to the technique: cost minimization in the choice of method. (iii) The second distinctive feature of the surplus approach is to take one distributive variable (usually w) as given for the determination of the surplus (and then of r or w).

Chapter 4

Sraffa's Theory of Prices

4.1 Mathematical Tool Set

Notation 1. Bold-faced letters represent vectors (lower-case) and matrices (upper-case), and scalars are represented by regular characters. We write $\mathcal{N} \equiv \{1, 2, \dots, n\}$ for the index set. The symbols a_i and b_j denote the i -th and j -th coordinates of column vectors \mathbf{a} and \mathbf{b}^T , respectively. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$: $\mathbf{a} \geq \mathbf{b}$ means $a_i \geq b_i \forall i \in \mathcal{N}$; $\mathbf{a} > \mathbf{b}$ means $a_i > b_i \forall i \in \mathcal{N}$. We write $\mathbb{R}_+^n \equiv \{\mathbf{a} \in \mathbb{R}^n : \mathbf{a} \geq \mathbf{0}\}$, $\mathbb{R}_+^n \setminus \{\mathbf{0}\} \equiv \{\mathbf{a} \in \mathbb{R}^n : \mathbf{a} > \mathbf{0}\}$ and $\mathbb{R}_{++}^n \equiv \{\mathbf{a} \in \mathbb{R}^n : \mathbf{a} > \mathbf{0}\}$, with analogous notation for matrices ($\mathbb{R}_+^{n \times n}$, $\mathbb{R}_{++}^{n \times n}$) and for row vectors ($\mathbb{R}_+^{1 \times n}$, $\mathbb{R}_{++}^{1 \times n}$). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be any $n \times n$ matrix, with m_{ij} its (i, j) entry for $i, j \in \mathcal{N}$; $\mathbf{M}_{(i)}$ and $\mathbf{M}^{(j)}$ denote its i -th row and j -th column. $\hat{\mathbf{a}} \equiv \text{diag}\{a_i\}$ is the diagonal matrix formed from \mathbf{a} . The summation vector is $\mathbf{e} \equiv (1, 1, \dots, 1)^T \in \mathbb{R}^n$. The null vector and null matrix are $\mathbf{0}$ and \mathbf{O} , and $\mathbf{I} \equiv \text{diag}\{1, 1, \dots, 1\}$ is the identity.

Definition 4 (Non-negative matrices). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$. \mathbf{M} is *non-negative* if $\mathbf{M} \geq \mathbf{O}$.

Definition 5 (Reducibility, irreducibility). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be non-negative. \mathbf{M} is *reducible* if there exists a permutation matrix \mathbf{P} such that

$$\mathbf{P}^T \mathbf{M} \mathbf{P} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{O} & \mathbf{M}_{22} \end{bmatrix},$$

where \mathbf{M}_{11} and \mathbf{M}_{22} are square submatrices of order at least one. \mathbf{M} is *irreducible* (or *indecomposable*) if it is not reducible.

Definition 6 (Directed graph). A *directed graph* (digraph) is a pair $G = (V, E)$, where V is a finite set of *vertices* and $E \subseteq V \times V$ is a set of *arcs* (ordered pairs). An arc $(i, j) \in E$ is written $i \rightarrow j$ and read “ i points to j ”.

Definition 7 (Walk, path, cycle, length). A *directed walk* of length $k \geq 1$ from i to j in $G = (V, E)$ is a sequence (v_0, v_1, \dots, v_k) with $v_0 = i$, $v_k = j$, and $(v_{t-1} \rightarrow v_t) \in E$ for every $t \in \{1, \dots, k\}$. The walk is a *path* if v_0, \dots, v_k are pairwise distinct, and a *cycle* (closed walk) if $v_0 = v_k$.

Definition 8 (Reachability, strong connectivity). Vertex j is *reachable* from i , written $i \Rightarrow j$, if G contains a directed walk from i to j (any walk reduces to a path by excising repeated vertices, so “walk” and “path” coincide here). G is *strongly connected* if $i \Rightarrow j$ for every ordered pair $(i, j) \in V \times V$.

Definition 9 (Period, aperiodicity). Let G be strongly connected with at least one arc. Its *period* is $h \equiv \gcd\{m \geq 1 : G \text{ contains a cycle of length } m\}$. G is *aperiodic* if $h = 1$.

Definition 10 (Primitive matrix). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be a non-negative matrix, \mathbf{M} is *primitive* if there exists $k \geq 1$ such that $\mathbf{M}^k > \mathbf{O}$ (such a matrix is, in particular, irreducible).

Proposition 1 (Powers count walks). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be non-negative and let $G(\mathbf{M})$ be its associated digraph, i.e. $i \rightarrow j$ if and only if $m_{ij} > 0$. For every $k \geq 1$ and every pair (i, j) , $(\mathbf{M}^k)_{ij} > 0$ if and only if $G(\mathbf{M})$ contains a directed walk of length k from i to j . Consequently: **(I)** $i \Rightarrow j \iff (\mathbf{M}^k)_{ij} > 0$ for some $k \in \{1, \dots, n\}$; **(II)** \mathbf{M} is irreducible $\iff G(\mathbf{M})$ is strongly connected $\iff (\mathbf{I} + \mathbf{M})^{n-1} > \mathbf{O}$; **(III)** \mathbf{M} is primitive $\iff G(\mathbf{M})$ is strongly connected and aperiodic.

Proof. See Meyer (2023). □

Definition 11 (Eigenvalues, eigenvectors, spectrum). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$. A scalar $\lambda \in \mathbb{C}$ and a nonzero vector $\mathbf{q} \in \mathbb{C}^n$ satisfying $\mathbf{M}\mathbf{q} = \lambda\mathbf{q}$ are called, respectively, an *eigenvalue* and an associated *eigenvector* of \mathbf{M} . Equivalently, λ is an eigenvalue of \mathbf{M} if and only if $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$, i.e. $\mathbf{M} - \lambda\mathbf{I}$ is singular. The *spectrum* of \mathbf{M} is the set of distinct eigenvalues $\sigma(\mathbf{M}) \equiv \{\lambda \in \mathbb{C} : \det(\mathbf{M} - \lambda\mathbf{I}) = 0\}$.

Definition 12 (Spectral radius). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$. The scalar $\rho(\mathbf{M}) \equiv \max_{\lambda \in \sigma(\mathbf{M})} |\lambda|$ is the *spectral radius* of \mathbf{M} .

Definition 13 (Multiplicities). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be a non-negative matrix, let $\lambda \in \sigma(\mathbf{M}) = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be the spectrum of \mathbf{M} . Then, $\forall \lambda_i \in \sigma(\mathbf{M})$ we define $\alpha(\lambda_i) = a_i$ as the *algebraic multiplicity* of λ_i , i.e., the number of times it is repeated as a root of the characteristic polynomial $\det(\mathbf{M} - \lambda \mathbf{I})$. If $\alpha(\lambda_i) = 1$, then λ_i is called a *simple eigenvalue*.

Proposition 2 (Spectral characterization of primitivity). *A non-negative irreducible matrix \mathbf{M} is primitive if and only if $\rho(\mathbf{M})$ is a simple and unique eigenvalue of maximal modulus, i.e., $|\lambda| < \rho(\mathbf{M})$ for every $\lambda \in \sigma(\mathbf{M}) \setminus \{\rho(\mathbf{M})\}$.*

Definition 14 (Collatz–Wielandt function). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be a non-negative matrix. The (primal) *Collatz–Wielandt function* of \mathbf{M} is $f : \mathbb{R}_+^n \setminus \{\mathbf{0}\} \rightarrow \mathbb{R}$,

$$f(\mathbf{x}) \equiv \min_{\substack{i \in \mathcal{N} \\ x_i \neq 0}} \frac{(\mathbf{M}\mathbf{x})_i}{x_i},$$

Definition 15 (Dual Collatz–Wielandt function). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be a non-negative matrix. The *dual Collatz–Wielandt function* of \mathbf{M} is $g : \mathbb{R}_{++}^n \rightarrow \mathbb{R}$,

$$g(\mathbf{x}) \equiv \max_{i \in \mathcal{N}} \frac{(\mathbf{M}\mathbf{x})_i}{x_i}.$$

Theorem 1 (Perron–Frobenius). *Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be a non-negative and irreducible matrix, let $\lambda^* \equiv \rho(\mathbf{M})$ be the spectral radius of \mathbf{M} . Then:*

(I) $\lambda^* > 0$ and $\lambda^* \in \sigma(\mathbf{M})$; it is called the *Perron-Frobenius eigenvalue* of \mathbf{M} ;

(II) λ^* is a *simple eigenvalue* of $\det(\mathbf{M} - \lambda \mathbf{I})$.

(III) *There exists a right eigenvector $\mathbf{q}_{\mathcal{R}} > \mathbf{0}$ with $\mathbf{M}\mathbf{q}_{\mathcal{R}} = \lambda^* \mathbf{q}_{\mathcal{R}}$; moreover λ^* is the only eigenvalue of \mathbf{M} admitting a non-negative eigenvector, and $\mathbf{q}_{\mathcal{R}}$ is unique up to a positive scalar.*

(IV) *There exists a left eigenvector $\mathbf{q}_{\mathcal{L}} > \mathbf{0}$ with $\mathbf{q}_{\mathcal{L}}^T \mathbf{M} = \lambda^* \mathbf{q}_{\mathcal{L}}^T$, unique up to a positive scalar.*

(V) (Collatz–Wielandt) the Perron–Frobenius eigenvalue is attained as

$$\lambda^* = \max_{\mathbf{x} \geq \mathbf{0}} f(\mathbf{x}) = \max_{\mathbf{x} \geq \mathbf{0}} \min_{\substack{i \in \mathcal{N} \\ x_i \neq 0}} \frac{(\mathbf{M}\mathbf{x})_i}{x_i},$$

the maximum being attained at $\mathbf{x} = \mathbf{q}_{\mathcal{R}}$, where $f(\mathbf{q}_{\mathcal{R}}) = \lambda^*$. (V') (Dual Collatz–Wielandt) the Perron–Frobenius eigenvalue is equally characterized as

$$\lambda^* = \min_{\mathbf{x} > \mathbf{0}} g(\mathbf{x}) = \min_{\mathbf{x} > \mathbf{0}} \max_{i \in \mathcal{N}} \frac{(\mathbf{M}\mathbf{x})_i}{x_i},$$

the minimum being attained at $\mathbf{x} = \mathbf{q}_{\mathcal{R}}$, where $g(\mathbf{q}_{\mathcal{R}}) = \lambda^*$. If, in addition, \mathbf{M} is primitive (Definition 10), then:

(VI) λ^* is the unique eigenvalue of maximal modulus: $|\lambda| < \lambda^*$ for every $\lambda \in \sigma(\mathbf{M}) \setminus \{\lambda^*\}$;

(VII) $\lim_{k \rightarrow \infty} \left(\frac{1}{\lambda^*} \mathbf{M} \right)^k = \frac{\mathbf{q}_{\mathcal{R}} \mathbf{q}_{\mathcal{L}}^T}{\mathbf{q}_{\mathcal{L}}^T \mathbf{q}_{\mathcal{R}}} > \mathbf{O}$.

Proof. See Meyer (2023, ch. 8). □

Corollary 1 (Strict Collatz–Wielandt comparison). Let $\mathbf{M} \in \mathbb{R}_+^{n \times n}$ be irreducible, let $\mathbf{x} > \mathbf{0}$ and $c \in \mathbb{R}$. (I) If $\mathbf{M}\mathbf{x} \geq c\mathbf{x}$ with $\mathbf{M}\mathbf{x} \neq c\mathbf{x}$, then $\rho(\mathbf{M}) > c$. (II) If $\mathbf{M}\mathbf{x} \leq c\mathbf{x}$ with $\mathbf{M}\mathbf{x} \neq c\mathbf{x}$, then $\rho(\mathbf{M}) < c$.

Proof. (I) Let $\mathbf{q}_{\mathcal{L}} > \mathbf{0}$ be the left Perron–Frobenius eigenvector of \mathbf{M} (Theorem 1, IV), $\mathbf{q}_{\mathcal{L}}^T \mathbf{M} = \rho(\mathbf{M}) \mathbf{q}_{\mathcal{L}}^T$. Then

$$\mathbf{q}_{\mathcal{L}}^T (\mathbf{M}\mathbf{x} - c\mathbf{x}) = \mathbf{q}_{\mathcal{L}}^T \mathbf{M}\mathbf{x} - c \mathbf{q}_{\mathcal{L}}^T \mathbf{x} = (\rho(\mathbf{M}) - c) \mathbf{q}_{\mathcal{L}}^T \mathbf{x}.$$

The left side is strictly positive: $\mathbf{M}\mathbf{x} - c\mathbf{x} \geq \mathbf{0}$ is nonzero and is contracted against $\mathbf{q}_{\mathcal{L}} > \mathbf{0}$. Since $\mathbf{q}_{\mathcal{L}}^T \mathbf{x} > 0$, dividing gives $\rho(\mathbf{M}) > c$. (II) Apply (I) to the reversed inequality, or repeat the computation with the sign of $\mathbf{M}\mathbf{x} - c\mathbf{x} \leq \mathbf{0}$ reversed. □

Proposition 3 (Position of the Perron–Frobenius eigenvalue relative to unity). Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be a non-negative and irreducible matrix, with a Perron–Frobenius eigenvalue $\lambda^* = \rho(\mathbf{M})$ and right Perron–Frobenius eigenvector $\mathbf{q}_{\mathcal{R}} > \mathbf{0}$. We have the following trichotomies hold.

(I) (Fixed-point face):

$$\lambda^* < 1 \iff \exists \mathbf{x} > \mathbf{0} : \mathbf{M}\mathbf{x} < \mathbf{x};$$

$$\lambda^* = 1 \iff \exists \mathbf{x} > \mathbf{0} : \mathbf{M}\mathbf{x} = \mathbf{x};$$

$$\lambda^* > 1 \iff \exists \mathbf{x} > \mathbf{0} : \mathbf{M}\mathbf{x} > \mathbf{x}.$$

(II) (Resolvent face) *The Neumann series $\sum_{k=0}^{\infty} \mathbf{M}^k$ converges if and only if $\lambda^* < 1$, in which case $\mathbf{I} - \mathbf{M}$ is non-singular and*

$$(\mathbf{I} - \mathbf{M})^{-1} = \sum_{k=0}^{\infty} \mathbf{M}^k > \mathbf{O}$$

(strictly positive, since irreducibility makes the digraph strongly connected, so for every ordered pair (i, j) some \mathbf{M}^k has a positive (i, j) entry). For $\lambda^ \geq 1$ the series diverges—since $\rho(\mathbf{M}^k) = (\lambda^*)^k \geq 1$ forbids $\mathbf{M}^k \rightarrow \mathbf{O}$ —and the non-negative inverse never exists: $\mathbf{I} - \mathbf{M}$ is singular precisely when $1 \in \sigma(\mathbf{M})$ (in particular always at $\lambda^* = 1$), and when $\lambda^* > 1$ with $1 \notin \sigma(\mathbf{M})$ the inverse exists but has at least one negative entry. By irreducibility, the partial sums diverge entrywise to $+\infty$; for primitive \mathbf{M} at $\lambda^* = 1$ they grow linearly, $\sum_{k=0}^K \mathbf{M}^k \sim (K + 1) \mathbf{q}_R \mathbf{q}_L^T / \mathbf{q}_L^T \mathbf{q}_R$.*

(III) (Dynamical face; \mathbf{M} primitive)

$$\begin{aligned} \lambda^* < 1 &\iff \mathbf{M}^k \rightarrow \mathbf{O} \\ \lambda^* = 1 &\iff \mathbf{M}^k \rightarrow \frac{\mathbf{q}_R \mathbf{q}_L^T}{\mathbf{q}_L^T \mathbf{q}_R} \\ \lambda^* > 1 &\iff \mathbf{M}^k \rightarrow \infty \text{ entrywise} \end{aligned}$$

as $k \rightarrow \infty$, where $\mathbf{q}_L > \mathbf{0}$ is the left Perron–Frobenius eigenvector.

Proof. (I) The forward implications take $\mathbf{x} = \mathbf{q}_R$, giving $\mathbf{M}\mathbf{q}_R = \lambda^*\mathbf{q}_R$, whence $\mathbf{M}\mathbf{q}_R \leq \mathbf{q}_R$ according as $\lambda^* \leq 1$. The reverse implications use both Collatz–Wielandt characterizations on a witness $\mathbf{x} > \mathbf{0}$ (all coordinates nonzero). If $\mathbf{M}\mathbf{x} > \mathbf{x}$ then $f(\mathbf{x}) > 1$, so by the max-form (Theorem 1, V) $\lambda^* \geq f(\mathbf{x}) > 1$. If $\mathbf{M}\mathbf{x} < \mathbf{x}$ then $g(\mathbf{x}) < 1$, so by the dual min-form (Theorem 1, V') $\lambda^* \leq g(\mathbf{x}) < 1$. If $\mathbf{M}\mathbf{x} = \mathbf{x}$ then $\mathbf{x} > \mathbf{0}$ is a positive eigenvector for the eigenvalue 1, and by Theorem 1 (III) the only eigenvalue admitting a non-negative eigenvector is λ^* , whence $\lambda^* = 1$. (II) The Neumann series $\sum_{k \geq 0} \mathbf{M}^k$ converges to $(\mathbf{I} - \mathbf{M})^{-1}$ if and only if $\rho(\mathbf{M}) < 1$; the sum is non-negative termwise and, by irreducibility, strictly positive (every ordered pair is joined by a walk, so some \mathbf{M}^k contributes a positive entry). This is the forward equivalence. For $\lambda^* \geq 1$: $\mathbf{I} - \mathbf{M}$ is singular if and only if $1 \in \sigma(\mathbf{M})$, which holds at $\lambda^* = 1$. Suppose now $\lambda^* > 1$, $1 \notin \sigma(\mathbf{M})$, so $(\mathbf{I} - \mathbf{M})^{-1}$ exists; we show it cannot be non-negative. Let $\mathbf{q}_R > \mathbf{0}$ be the right Perron–Frobenius eigenvector, $\mathbf{M}\mathbf{q}_R = \lambda^*\mathbf{q}_R$, so

$(\mathbf{I} - \mathbf{M})\mathbf{q}_{\mathcal{R}} = (1 - \lambda^*)\mathbf{q}_{\mathcal{R}}$ and hence

$$(\mathbf{I} - \mathbf{M})^{-1}\mathbf{q}_{\mathcal{R}} = \frac{1}{1 - \lambda^*}\mathbf{q}_{\mathcal{R}} < \mathbf{0},$$

the scalar being negative since $\lambda^* > 1$ and $\mathbf{q}_{\mathcal{R}} > \mathbf{0}$. Were $(\mathbf{I} - \mathbf{M})^{-1} \geq \mathbf{O}$, then $(\mathbf{I} - \mathbf{M})^{-1}\mathbf{q}_{\mathcal{R}} \geq \mathbf{0}$, a contradiction. Thus $(\mathbf{I} - \mathbf{M})^{-1}$ has a negative entry. **(III)** By Theorem 1, (VII), $(\mathbf{M}/\lambda^*)^k \rightarrow \mathbf{q}_{\mathcal{R}}\mathbf{q}_{\mathcal{L}}^T/\mathbf{q}_{\mathcal{L}}^T\mathbf{q}_{\mathcal{R}}$, so $\mathbf{M}^k \sim \lambda^{*k}\mathbf{q}_{\mathcal{R}}\mathbf{q}_{\mathcal{L}}^T/\mathbf{q}_{\mathcal{L}}^T\mathbf{q}_{\mathcal{R}}$; the three regimes follow from $\lambda^{*k} \rightarrow 0, 1, \infty$. \square

4.2 Preliminary Definitions

Assumption 2 (Set I). The economic system of this chapter rests on the following assumptions, unless indicated otherwise:

- (\mathfrak{A}_1) the economic system is in a perfectly stationary state;
- (\mathfrak{A}_2) each production process yields exactly one commodity (no joint production);
- (\mathfrak{A}_3) there is no fixed capital;
- (\mathfrak{A}_4) labour is homogeneous and indispensable.

Definition 16 (Production of commodities by means of commodities). Let $\mathcal{N} = \{1, 2, \dots, n\}$ index the commodities. For each $j \in \mathcal{N}$ the observed process producing commodity j consumes a quantity $z_{ij} \geq 0$ of each commodity $i \in \mathcal{N}$ together with a quantity $L_j > 0$ of labour, and yields an observed gross output $x_j > 0$:

$$\left(\bigoplus_{i \in \mathcal{N}} z_{ij} \right) \oplus L_j \longrightarrow x_j,$$

where \oplus informally denotes the ‘‘combination’’ of inputs within a single process. Commodities are thus produced by means of commodities, including themselves (z_{ii} may be positive), and of labour.

Definition 17 (Technical and labour coefficients). For each ordered pair $(i, j) \in \mathcal{N} \times \mathcal{N}$ define the *technical coefficient* $a_{ij} \equiv \frac{z_{ij}}{x_j}$, the quantity of commodity i required per unit of

commodity j , and the *labour coefficient* $l_j \equiv \frac{L_j}{x_j}$, the labour required per unit of commodity j . The unit process for commodity j is

$$\left(\bigoplus_{i \in \mathcal{N}} a_{ij} \right) \oplus l_j \longrightarrow 1 \text{ unit of } j.$$

Each coefficient records a fixed input proportion at the observed output level; it carries no claim of invariance to scale (see Remark 1). (Miller & Blair, 2009, p. 16)

4.2.1 The Quantity system

Definition 18 (Quantity system). Let $\mathbf{x} = \{x_j\} \in \mathbb{R}_{++}^n$ be the *gross-output vector*. Let $\mathbf{Z} = \{z_{ij}\} \in \mathbb{R}_+^{n \times n}$ and $\mathbf{A} = \{a_{ij}\} \in \mathbb{R}_+^{n \times n}$ (with $\mathbf{A} \equiv \mathbf{Z}\hat{\mathbf{x}}^{-1}$) be the matrices of *means of production* and *technical coefficients*. Let $\boldsymbol{\mu} = \{L_j\} \in \mathbb{R}_{++}^n$ and $\mathbf{l} = \{l_j\} \in \mathbb{R}_{++}^n$ (with $\mathbf{l} \equiv \hat{\mathbf{x}}^{-1}\boldsymbol{\mu}$) be the vectors of *labour inputs* and *labour coefficients*. Let $\mathbf{s} = \mathbf{A}\mathbf{x} = \mathbf{Z}\mathbf{e}$ be the vector of *stocks* of means of production, we define

$$\mathbf{y} \equiv (\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{x} - \mathbf{s}, \quad (4.1)$$

as the technical net product (surplus), i.e., output after the substitution of intermediate inputs. The set $\mathcal{T} = \{\mathbf{A}, \mathbf{l}, \mathbf{x}\}$ characterizes the *technique* of production of the economy. Moreover, if $\rho(\mathbf{A}) < 1$, by Proposition 3 matrix $\mathbf{I} - \mathbf{A}$ is non-singular and non-negative, hence we can write

$$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} \quad (4.2)$$

Remark 1 (Fixed proportions are not constant returns). Definition 17 fixes the *proportions* among inputs. Treating \mathbf{A} as valid at all output levels would add the distinct assumption of constant returns to scale. Following Sraffa (1960), we do not require it: the coefficients are read off a single observed position of the economy (as *data*), and no variation of x_j is contemplated. Sraffa's quantity relation (see 18), though algebraically identical to the open Leontief system $\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{y}$ (ch. IV Pasinetti, 1977, see), differs in one crucial aspect: in the open Leontief system \mathbf{y} is data and $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}$ is solved *counter-factually* (which presupposes constant returns), whereas here \mathbf{x} is observed, and the unknowns are prices and distribution.

Definition 19 (Subsistence vector, subsistence matrix). Let $\boldsymbol{\gamma} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ be the vector of commodities that one unit of labor needs for subsistence, i.e., the *subsistence vector*. Define $\boldsymbol{\Gamma} = \{\Gamma_{ij}\} \in \mathbb{R}_+^{n \times n}$ as the *subsistence matrix* (with $\boldsymbol{\Gamma} \equiv \boldsymbol{\gamma}\boldsymbol{\mu}^T$). Where each entry Γ_{ij} is the amount of commodity i that the workers of industry j consume.

Definition 20 (Socio-technical means of production and coefficients matrices, socio-technical vector of stocks of mean of production). We define $\tilde{\mathbf{Z}} = \{\tilde{z}_{ij}\} \in \mathbb{R}_+^{n \times n}$ (with $\tilde{\mathbf{Z}} \equiv \mathbf{Z} + \boldsymbol{\Gamma}$) and $\tilde{\mathbf{A}} = \{\tilde{a}_{ij}\} \in \mathbb{R}_+^{n \times n}$ (with $\tilde{\mathbf{A}} \equiv \tilde{\mathbf{Z}}\hat{\mathbf{x}}^{-1}$) as the matrices of *socio-technical means of production* and *coefficients*. Then $\tilde{\mathbf{s}} = \tilde{\mathbf{A}}\mathbf{x}$ is the vector of stocks of *socio-technical* means of production.

4.2.2 Conditions for a Surplus

Proposition 4 (Monotonicity of the Perron–Frobenius eigenvalue). *Let $\mathbf{A}, \tilde{\mathbf{A}} \in \mathbb{R}_+^{n \times n}$ be irreducible with $\mathbf{A} \leq \tilde{\mathbf{A}}$. Then $\rho(\mathbf{A}) \leq \rho(\tilde{\mathbf{A}})$, with equality if and only if $\mathbf{A} = \tilde{\mathbf{A}}$.*

Proof. Weak inequality. Let $\mathbf{q}_{\mathcal{R}} > \mathbf{0}$ be the right Perron–Frobenius eigenvector of \mathbf{A} (Theorem 1, III), so $\mathbf{A}\mathbf{q}_{\mathcal{R}} = \rho(\mathbf{A})\mathbf{q}_{\mathcal{R}}$. Since $\mathbf{A} \leq \tilde{\mathbf{A}}$ and $\mathbf{q}_{\mathcal{R}} > \mathbf{0}$, $\tilde{\mathbf{A}}\mathbf{q}_{\mathcal{R}} \geq \mathbf{A}\mathbf{q}_{\mathcal{R}} = \rho(\mathbf{A})\mathbf{q}_{\mathcal{R}}$. Reading this last expression coordinate by coordinate, $(\tilde{\mathbf{A}}\mathbf{q}_{\mathcal{R}})_i / (\mathbf{q}_{\mathcal{R}})_i \geq \rho(\mathbf{A})$ for every i , hence $f_{\tilde{\mathbf{A}}}(\mathbf{q}_{\mathcal{R}}) = \min_i (\tilde{\mathbf{A}}\mathbf{q}_{\mathcal{R}})_i / (\mathbf{q}_{\mathcal{R}})_i \geq \rho(\mathbf{A})$. By the Collatz–Wielandt characterization of $\rho(\tilde{\mathbf{A}})$ as a maximum of $f_{\tilde{\mathbf{A}}}$ (Theorem 1, V), $\rho(\tilde{\mathbf{A}}) = \max_{\mathbf{x} \geq \mathbf{0}} f_{\tilde{\mathbf{A}}}(\mathbf{x}) \geq f_{\tilde{\mathbf{A}}}(\mathbf{q}_{\mathcal{R}}) \geq \rho(\mathbf{A})$. *Equality case.* If $\mathbf{A} = \tilde{\mathbf{A}}$ then trivially $\rho(\mathbf{A}) = \rho(\tilde{\mathbf{A}})$. Conversely, suppose $\mathbf{A} \not\leq \tilde{\mathbf{A}}$. With $\mathbf{q}_{\mathcal{R}} > \mathbf{0}$ the right Perron–Frobenius eigenvector of \mathbf{A} , the slack $(\tilde{\mathbf{A}} - \mathbf{A})\mathbf{q}_{\mathcal{R}} \geq \mathbf{0}$ is nonzero, since $\tilde{\mathbf{A}} - \mathbf{A} \geq \mathbf{0}$ is nonzero and $\mathbf{q}_{\mathcal{R}} > \mathbf{0}$. Hence

$$\tilde{\mathbf{A}}\mathbf{q}_{\mathcal{R}} = \mathbf{A}\mathbf{q}_{\mathcal{R}} + (\tilde{\mathbf{A}} - \mathbf{A})\mathbf{q}_{\mathcal{R}} = \rho(\mathbf{A})\mathbf{q}_{\mathcal{R}} + (\tilde{\mathbf{A}} - \mathbf{A})\mathbf{q}_{\mathcal{R}} \geq \rho(\mathbf{A})\mathbf{q}_{\mathcal{R}}, \quad \tilde{\mathbf{A}}\mathbf{q}_{\mathcal{R}} \neq \rho(\mathbf{A})\mathbf{q}_{\mathcal{R}}.$$

Applying Corollary 1 (I) to the irreducible matrix $\tilde{\mathbf{A}}$, with $\mathbf{x} = \mathbf{q}_{\mathcal{R}} > \mathbf{0}$ and $c = \rho(\mathbf{A})$, yields $\rho(\tilde{\mathbf{A}}) > \rho(\mathbf{A})$. \square

Lemma 1 (Value of the net product at Perron weights). *Let $\mathbf{A} \in \mathbb{R}_+^{n \times n}$ be irreducible, with Perron–Frobenius eigenvalue $\lambda^* = \rho(\mathbf{A})$ and left Perron–Frobenius eigenvector $\mathbf{q}_{\mathcal{L}} > \mathbf{0}$ (Theorem 1, IV). For every $\mathbf{x} > \mathbf{0}$, the net product $\mathbf{y} = (\mathbf{I} - \mathbf{A})\mathbf{x}$ satisfies $\mathbf{q}_{\mathcal{L}}^T \mathbf{y} = (1 - \lambda^*)\mathbf{q}_{\mathcal{L}}^T \mathbf{x}$. Since $\mathbf{q}_{\mathcal{L}}^T \mathbf{x} > 0$, the scalar $\mathbf{q}_{\mathcal{L}}^T \mathbf{y}$ has the sign of $1 - \lambda^*$: $\mathbf{q}_{\mathcal{L}}^T \mathbf{y} > 0 \iff \lambda^* < 1$; $\mathbf{q}_{\mathcal{L}}^T \mathbf{y} = 0 \iff$*

$\lambda^* = 1$; and $\mathbf{q}_{\mathcal{L}}^T \mathbf{y} < 0 \iff \lambda^* > 1$. In particular, no $\mathbf{x} > \mathbf{0}$ can yield a semipositive $\mathbf{y} \geq \mathbf{0}$ unless $\lambda^* \leq 1$, and none can yield a strictly positive $\mathbf{y} > \mathbf{0}$ unless $\lambda^* < 1$.

Proof. Expanding and using $\mathbf{q}_{\mathcal{L}}^T \mathbf{A} = \lambda^* \mathbf{q}_{\mathcal{L}}^T$, $\mathbf{q}_{\mathcal{L}}^T (\mathbf{I} - \mathbf{A}) \mathbf{x} = (1 - \lambda^*) \mathbf{q}_{\mathcal{L}}^T \mathbf{x}$, and $\mathbf{q}_{\mathcal{L}}^T \mathbf{x} > 0$ because $\mathbf{q}_{\mathcal{L}} > \mathbf{0}$ and $\mathbf{x} > \mathbf{0}$; the sign equivalences follow. For the last claim: if $\mathbf{y} \geq \mathbf{0}$ then $\mathbf{q}_{\mathcal{L}}^T \mathbf{y} \geq 0$ (with $\mathbf{q}_{\mathcal{L}} > \mathbf{0}$), forcing $1 - \lambda^* \geq 0$; and if $\mathbf{y} > \mathbf{0}$ then $\mathbf{q}_{\mathcal{L}}^T \mathbf{y} > 0$, forcing $1 - \lambda^* > 0$. \square

Proposition 5 (Surplus conditions and their economic content). *Let \mathbf{A} be the technical and $\tilde{\mathbf{A}} = \mathbf{A} + \gamma \mathbf{l}^T$ the socio-technical coefficient matrix, both irreducible, with $\gamma \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ and $\mathbf{l} > \mathbf{0}$. Write $\lambda^* = \rho(\mathbf{A})$, $\tilde{\lambda}^* = \rho(\tilde{\mathbf{A}})$, with right and left Perron–Frobenius eigenvectors $\mathbf{q}_{\mathcal{R}}, \mathbf{q}_{\mathcal{L}} > \mathbf{0}$ (Theorem 1). Each face of Proposition 3, read on \mathbf{A} , classifies the economy by its net product; read on $\tilde{\mathbf{A}}$, by its surplus over subsistence.*

(I) (Fixed-point: self-replacement and the net product.) *For $\mathbf{x} > \mathbf{0}$,*

$$\begin{aligned} \lambda^* < 1 &\iff \exists \mathbf{x} > \mathbf{0} : \mathbf{A}\mathbf{x} < \mathbf{x} && \text{(inputs reproduced with room to spare),} \\ \lambda^* = 1 &\iff \exists \mathbf{x} > \mathbf{0} : \mathbf{A}\mathbf{x} = \mathbf{x} && \text{(bare self-replacement),} \\ \lambda^* > 1 &\iff \exists \mathbf{x} > \mathbf{0} : \mathbf{A}\mathbf{x} > \mathbf{x} && \text{(no self-replacing scale).} \end{aligned}$$

At the Perron scale $\mathbf{x} = \mathbf{q}_{\mathcal{R}}$ the net product is $\mathbf{y} = (\mathbf{I} - \mathbf{A})\mathbf{q}_{\mathcal{R}} = (1 - \lambda^)\mathbf{q}_{\mathcal{R}}$, so $\mathbf{y} > \mathbf{0}$, $\mathbf{y} = \mathbf{0}$, or $\mathbf{y} < \mathbf{0}$ according as $\lambda^* \leq 1$; for an arbitrary $\mathbf{x} > \mathbf{0}$ only the aggregate $\mathbf{q}_{\mathcal{L}}^T \mathbf{y} = (1 - \lambda^*)\mathbf{q}_{\mathcal{L}}^T \mathbf{x}$ is signed (Lemma 1). Hence, a semipositive net product, $\mathbf{y} \geq \mathbf{0}$, is attainable only for $\lambda^* < 1$: at $\lambda^* = 1$ the unique self-replacing scale leaves $\mathbf{y} = \mathbf{0}$. Read on $\tilde{\mathbf{A}}$: a scale at which the means of production and workers' subsistence are jointly reproduced exists iff $\tilde{\lambda}^* \leq 1$, with a surplus over subsistence in every commodity iff $\tilde{\lambda}^* < 1$.*

(II) (Resolvent: the Leontief multiplier.) *The dated expansion $\sum_{k \geq 0} \mathbf{A}^k$ converges iff $\lambda^* < 1$, in which case $(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k \geq 0} \mathbf{A}^k > \mathbf{0}$ (strictly, by irreducibility). Economically, $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{y}$ is the gross output that delivers a prescribed net product $\mathbf{y} \geq \mathbf{0}$, resolved into the direct requirements \mathbf{y} , the inputs of those inputs $\mathbf{A}\mathbf{y}$, the inputs of those $\mathbf{A}^2 \mathbf{y}$, and so on—the vertically integrated (“dated”) requirements, finite precisely when $\lambda^* < 1$. For $\lambda^* \geq 1$ the series diverges—no finite gross output sustains a positive net product—and the non-negative inverse fails: $\mathbf{I} - \mathbf{A}$ is singular at $\lambda^* = 1$, and for $\lambda^* > 1$ the inverse, where it exists, carries a negative entry. This is the algebraic counterpart of productivity.*

(III) (Dynamical: viability of repeated production; \mathbf{A} primitive.) With $\mathbf{A}^k \mathbf{x}$ the input bundle supporting production k stages deep,

$$\lambda^* < 1 \iff \mathbf{A}^k \rightarrow \mathbf{O} \quad (\text{indirect requirements vanish; a surplus remains}),$$

$$\lambda^* = 1 \iff \mathbf{A}^k \rightarrow \frac{\mathbf{q}_{\mathcal{R}} \mathbf{q}_{\mathcal{L}}^T}{\mathbf{q}_{\mathcal{L}}^T \mathbf{q}_{\mathcal{R}}} \quad (\text{inputs reproduce period by period, nothing to spare}),$$

$$\lambda^* > 1 \iff \mathbf{A}^k \rightarrow \infty \text{ entrywise} \quad (\text{requirements explode; no self-sustainment}),$$

as $k \rightarrow \infty$, the limit at $\lambda^* = 1$ being the Perron–Frobenius projector.

(IV) (Necessity, not sufficiency.) Since $\tilde{\mathbf{A}} - \mathbf{A} = \gamma \mathbf{l}^T \geq \mathbf{O}$ is nonzero ($\gamma \neq \mathbf{0}, \mathbf{l} > \mathbf{0}$), Proposition 4 yields the strict inequality $\lambda^* < \tilde{\lambda}^*$. Hence $\tilde{\lambda}^* \leq 1$ forces $\lambda^* < \tilde{\lambda}^* \leq 1$: a strict physical surplus over the means of production, $\lambda^* < 1$, is necessary for the joint reproduction of means of production and subsistence, but not sufficient— $\lambda^* < 1$ is compatible with $\tilde{\lambda}^* > 1$ once γ is large enough, since scaling γ upward drives $\tilde{\lambda}^* \rightarrow \infty$ while leaving λ^* fixed. An economy may produce a physical surplus over its means of production and still fails to provide for the subsistence of its labour force.

Proof. (I)–(III) are the three faces of Proposition 3 applied to \mathbf{A} and to $\tilde{\mathbf{A}}$, with the net product signed by Lemma 1: at $\mathbf{x} = \mathbf{q}_{\mathcal{R}}, \mathbf{y} = (1 - \lambda^*) \mathbf{q}_{\mathcal{R}}$, and any semipositive $\mathbf{y} \geq \mathbf{0}$ gives $\mathbf{q}_{\mathcal{L}}^T \mathbf{y} > 0$, forcing $\lambda^* < 1$. (IV) is Proposition 4 with $\tilde{\mathbf{A}} - \mathbf{A} = \gamma \mathbf{l}^T \geq \mathbf{O}$ nonzero (so the inequality is strict), together with $\rho(\mathbf{A} + t \gamma \mathbf{l}^T) \geq t \mathbf{l}^T \gamma \rightarrow \infty$ as $t \rightarrow \infty$, which leaves $\lambda^* = \rho(\mathbf{A})$ untouched. \square

4.2.3 Basic and Non-basic Commodities

Definition 21 (Dependency digraph). The *dependency digraph* of \mathbf{A} is the digraph $G(\mathbf{A}) = (\mathcal{N}, E)$ associated to \mathbf{A} in the sense of Proposition 1, i.e. $(i \rightarrow j) \in E \iff a_{ij} > 0$. Commodity i enters directly or indirectly into commodity j if $i \Rightarrow j$ in $G(\mathbf{A})$ (Definition 8). By Proposition 1 (I), this holds if and only if $(\mathbf{A}^k)_{ij} > 0$ for some $k \in \{1, \dots, n\}$.

Definition 22 (Basic and non-basic commodities). Commodity i is *basic* if $i \Rightarrow j$ for every $j \in \mathcal{N}$, and *non-basic* otherwise. The *basic set* is $\mathcal{B} \equiv \{i \in \mathcal{N} : i \Rightarrow j \ \forall j \in \mathcal{N}\}$, with $\mathcal{B} \subseteq \mathcal{N}$. The *non-basic set* is $\mathcal{D} \equiv \mathcal{N} \setminus \mathcal{B}$. This implies that $\mathcal{N} = \mathcal{B} \cup \mathcal{D}$ and $\mathcal{B} \cap \mathcal{D} = \emptyset$.

Proposition 6 (Characterizations of an all-basic system). *The following are equivalent: (I) $\mathcal{B} = \mathcal{N}$ (every commodity is basic); (II) $G(\mathbf{A})$ is strongly connected; (III) \mathbf{A} is irreducible; (IV) $\sum_{k=1}^n \mathbf{A}^k > \mathbf{O}$.*

Proof. (I) \iff (II): by Definition 22, $\mathcal{B} = \mathcal{N}$ means $i \Rightarrow j$ for every ordered pair (i, j) , which is strong connectivity (Definition 8). (II) \iff (III): Proposition 1 (II). (I) \iff (IV): by Proposition 1 (I), $i \Rightarrow j$ if and only if $(\mathbf{A}^k)_{ij} > 0$ for some $k \leq n$, i.e. if and only if $(\sum_{k=1}^n \mathbf{A}^k)_{ij} > 0$ (a sum of non-negative terms is positive if and only if one term is); requiring this for all (i, j) is $\sum_{k=1}^n \mathbf{A}^k > \mathbf{O}$. \square

Proposition 7 (Block-triangular form). *Let $\emptyset \neq \mathcal{B} \subset \mathcal{N}$. Ordering the commodities so that those in \mathcal{B} precede those in $\mathcal{D} = \mathcal{N} \setminus \mathcal{B} \neq \emptyset$, there is a permutation matrix \mathbf{P} with*

$$\mathbf{P}^T \mathbf{A} \mathbf{P} = \begin{bmatrix} \mathbf{A}_{\mathcal{B}\mathcal{B}} & \mathbf{A}_{\mathcal{B}\mathcal{D}} \\ \mathbf{O} & \mathbf{A}_{\mathcal{D}\mathcal{D}} \end{bmatrix},$$

where the leading block $\mathbf{A}_{\mathcal{B}\mathcal{B}}$ is irreducible and the block $\mathbf{A}_{\mathcal{D}\mathcal{B}}$ vanishes.

Proof. Suppose, for contradiction, $a_{ij} > 0$ with $i \in \mathcal{D}$ and $j \in \mathcal{B}$. Then $i \rightarrow j$, and since j is basic, $j \Rightarrow m$ for every $m \in \mathcal{N}$; prepending the arc $i \rightarrow j$ gives $i \Rightarrow m$ for every m , so $i \in \mathcal{B}$, which contradicts $i \in \mathcal{D}$. Hence no arc runs from \mathcal{D} to \mathcal{B} , i.e. $\mathbf{A}_{\mathcal{D}\mathcal{B}} = \mathbf{O}$. Now, for the irreducibility of $\mathbf{A}_{\mathcal{B}\mathcal{B}}$: take any $i, j \in \mathcal{B}$. Since i is basic, $i \Rightarrow j$, so a path $i \rightarrow v_1 \rightarrow \dots \rightarrow j$ exists in $G(\mathbf{A})$. Every intermediate vertex v_t satisfies $v_t \Rightarrow j$ and, being reached from the basic vertex i , lies on a route to all of \mathcal{N} ; more directly, no arc leaves \mathcal{B} for \mathcal{D} and returns (as $\mathbf{A}_{\mathcal{D}\mathcal{B}} = \mathbf{O}$ forbids the return), so the path cannot exit \mathcal{B} and re-enter. Thus, the path stays within \mathcal{B} , giving reachability inside the subgraph induced on \mathcal{B} . Hence, that subgraph is strongly connected and, by Proposition 1 (II), $\mathbf{A}_{\mathcal{B}\mathcal{B}}$ is irreducible. \square

Corollary 2 (Basic commodities and the spectral radius). *Under Proposition 7, $\sigma(\mathbf{A}) = \sigma(\mathbf{A}_{\mathcal{B}\mathcal{B}}) \cup \sigma(\mathbf{A}_{\mathcal{D}\mathcal{D}}$ and $\rho(\mathbf{A}) = \max\{\rho(\mathbf{A}_{\mathcal{B}\mathcal{B}}), \rho(\mathbf{A}_{\mathcal{D}\mathcal{D}})\}$. If moreover $\rho(\mathbf{A}_{\mathcal{D}\mathcal{D}}) < \rho(\mathbf{A}_{\mathcal{B}\mathcal{B}})$, then $\rho(\mathbf{A})$ is the Perron–Frobenius eigenvalue of the irreducible basic block $\mathbf{A}_{\mathcal{B}\mathcal{B}}$, and Theorem 1 applies to $\mathbf{A}_{\mathcal{B}\mathcal{B}}$.*

Proof. The spectrum of a block-triangular matrix is the union of the spectra of its diagonal blocks. Under the stated inequality the maximum is $\rho(\mathbf{A}_{\mathcal{B}\mathcal{B}})$, which by Theorem 1 (I) is the simple, positive Perron–Frobenius eigenvalue of the irreducible block $\mathbf{A}_{\mathcal{B}\mathcal{B}}$. \square

4.3 Production Without a Surplus

Definition 23 (Subsistence economy). For a given economy let $\mathbf{A} \in \mathbb{R}_+^{n \times n}$ be the (irreducible) technical coefficient matrix and $\boldsymbol{\omega} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ the bundle of commodities the ruling wage w can purchase, i.e., $w = \mathbf{p}^T \boldsymbol{\omega}$. The economy is a *subsistence economy* (in the Sraffian sense) if the wage is at subsistence, $\boldsymbol{\omega} = \boldsymbol{\gamma}$, so that workers' wages are absorbed into the socio-technical matrix $\tilde{\mathbf{A}} \equiv \mathbf{A} + \boldsymbol{\gamma} \mathbf{l}^T$, and the latter is self-replacing: $\rho(\tilde{\mathbf{A}}) = 1$, equivalently, $\tilde{\mathbf{A}} \mathbf{x} = \mathbf{x}$, and hence ($\tilde{\mathbf{s}} = \mathbf{x}$): this implies that $\mathbf{x} = \mathbf{q}_{\mathcal{R}}$, i.e., \mathbf{x} is the right-hand Perron–Frobenius eigenvector of $\tilde{\mathbf{A}}$.

Remark 2 (A physical surplus over means of production still exists). By Proposition 4, $\rho(\mathbf{A}) < \rho(\tilde{\mathbf{A}}) = 1$, so by Lemma 1 the technical net product carries strictly positive value, $\mathbf{q}_{\mathcal{L}}^T \mathbf{y} = (1 - \rho(\mathbf{A})) \mathbf{q}_{\mathcal{L}}^T \mathbf{x} > 0$: the economy produces a surplus over its means of production. Evaluating at $\mathbf{x} = \mathbf{q}_{\mathcal{R}}$ and $\mathbf{l}^T \mathbf{x} = \mathbf{l}^T \mathbf{q}_{\mathcal{R}} = \sum_{j=1}^n L_j = L$,

$$\mathbf{y} = (\mathbf{I} - \mathbf{A}) \mathbf{q}_{\mathcal{R}} = (\mathbf{I} - \tilde{\mathbf{A}}) \mathbf{q}_{\mathcal{R}} + \boldsymbol{\gamma} \mathbf{l}^T \mathbf{q}_{\mathcal{R}} = \boldsymbol{\gamma} \mathbf{l}^T \mathbf{q}_{\mathcal{R}} = \boldsymbol{\gamma} L \geq \mathbf{0},$$

which is strictly positive in coordinate i exactly where $\gamma_i > 0$; thus $\mathbf{y} > \mathbf{0}$ if and only if $\boldsymbol{\gamma} > \mathbf{0}$. The technical net output equals the vector of commodities needed for the subsistence of the whole workforce.

Proposition 8 (Subsistence prices). Let $\tilde{\mathbf{A}} \in \mathbb{R}_+^{n \times n}$ be irreducible with $\rho(\tilde{\mathbf{A}}) = 1$, and let $\boldsymbol{\omega} = \boldsymbol{\gamma}$. The price system is a homogeneous relation

$$\mathbf{p}^T = \mathbf{p}^T \tilde{\mathbf{A}}. \quad (4.3)$$

The rate of profit of the economy is $r = 0$, necessarily. Moreover, the price vector is $\mathbf{p}^T = \mathbf{q}_{\mathcal{L}}^T > \mathbf{0}$, the left-hand Perron–Frobenius eigenvector of $\tilde{\mathbf{A}}$.

Proof. With $\rho(\tilde{\mathbf{A}}) = 1$, the relation $\mathbf{p}^T = \mathbf{p}^T \tilde{\mathbf{A}}$ states that \mathbf{p}^T is a left-hand eigenvector of $\tilde{\mathbf{A}}$ for the eigenvalue $1 = \rho(\tilde{\mathbf{A}})$. By Theorem 1 (IV), the only such vector with non-negative entries is the left-hand Perron–Frobenius eigenvector $\mathbf{q}_{\mathcal{L}}^T > \mathbf{0}$, unique up to scale; hence $\mathbf{p}^T = \mathbf{q}_{\mathcal{L}}^T$. That $r = 0$: any uniform profit factor $1 + r > 1$ would require $\mathbf{p}^T = (1 + r) \mathbf{p}^T \tilde{\mathbf{A}}$, i.e. $\mathbf{p}^T \tilde{\mathbf{A}} = (1 + r)^{-1} \mathbf{p}^T$, making $(1 + r)^{-1} < 1$ an eigenvalue of $\tilde{\mathbf{A}}^T$ associated to strictly

positive eigenvector $\mathbf{p}^T = \mathbf{q}_{\mathcal{L}}^T > \mathbf{0}$, however, this is a contradiction, since $\rho(\tilde{\mathbf{A}}) = 1$ is the only eigenvalue admitting a positive eigenvector (Theorem 1, III–IV). Equivalently, by Lemma 1, the net product has zero value at $\mathbf{q}_{\mathcal{L}}$, leaving nothing to fund a positive profit. \square

Definition 24 (Numéraire and normalization). Price equations pin down the price vector only as a *ray*: if \mathbf{p}^T is a solution, so is $\mu\mathbf{p}^T$ for every $\mu > 0$. Fixing a *numéraire*, i.e., a commodity bundle $\mathbf{d} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ chosen (not determined) as the unit of account, allows us to find a representative solution. The *normalization condition* is

$$\mathbf{p}^T \mathbf{d} = 1.$$

Let $\mathbf{p}_*^T > \mathbf{0}$ be *any* solution of the price equations and $\{\mu\mathbf{p}_*^T : \mu > 0\}$ its ray. We have that $\mathbf{p}_*^T \mathbf{d} > 0$, so imposing the normalization on a generic ray element yields a unique scale,

$$(\mu\mathbf{p}_*^T)\mathbf{d} = 1 \iff \mu = \frac{1}{\mathbf{p}_*^T \mathbf{d}} > 0.$$

Hence, the normalization is met by *exactly one* point of the ray, the *normalized price vector*

$$\hat{\mathbf{p}}^T \equiv \frac{\mathbf{p}_*^T}{\mathbf{p}_*^T \mathbf{d}}, \quad \hat{\mathbf{p}}^T \mathbf{d} = \frac{\mathbf{p}_*^T \mathbf{d}}{\mathbf{p}_*^T \mathbf{d}} = 1.$$

There is no circularity. The equation $\mathbf{p}^T \mathbf{d} = 1$ is not a property shared by all solutions but a rule that picks one of them; the fact that $\mathbf{p}_*^T / (\mathbf{p}_*^T \mathbf{d})$ is again a solution, which is exactly the point, since it is the one on which the rule bites. The numéraire fixes the scale only: the relative prices $\hat{p}_i / \hat{p}_j = p_{*,i} / p_{*,j}$ are invariant to both μ and the choice of \mathbf{d} .

4.4 Production With Surplus

Assumption 3 (Set II). The economic system of this section adds on the Assumptions made in 2: $\mathfrak{A}_1 - \mathfrak{A}_4$:

(\mathfrak{A}_5) Wages above subsistence are paid *post-factum*;

(\mathfrak{A}_6) $L = \mathbf{l}^T \mathbf{x} = 1$ (labour normalization);

(\mathfrak{A}_7) $Y = \mathbf{p}^T \mathbf{y} = 1$ (net product as numéraire).

(\mathfrak{A}_8) \mathbf{A} is irreducible (only basic commodities).

Definition 25. Let $\varphi = \frac{wL}{Y}$ be the wage share of the economy; then, under $\mathfrak{A}_6 - \mathfrak{A}_7$, the wage rate is equal to the wage share ($w = \varphi$).

Definition 26 (Economy with surplus). For a given economy let $\tilde{\mathbf{A}} \in \mathbb{R}_+^{n \times n}$ be its irreducible socio-technical coefficients matrix, let $\rho(\tilde{\mathbf{A}}) < 1$, which implies, by Lemma 1, that $\mathbf{y} \geq \mathbf{0}$.

Proposition 9 (Price system with surplus, subsistence wages). Let $\tilde{\mathbf{A}} \in \mathbb{R}_+^{n \times n}$ be irreducible with $\rho(\tilde{\mathbf{A}}) < 1$, and let $\boldsymbol{\omega} = \boldsymbol{\gamma}$. Then the price system is

$$\mathbf{p}^T = (1 + r) \mathbf{p}^T \tilde{\mathbf{A}}. \quad (4.4)$$

The uniform profit rate is uniquely determined,

$$r = \frac{1}{\rho(\tilde{\mathbf{A}})} - 1 > 0, \quad (4.5)$$

and $\mathbf{p}^T = \mathbf{q}_{\mathcal{L}}^T > \mathbf{0}$ is the left-hand Perron–Frobenius eigenvector of $\tilde{\mathbf{A}}$, fixed in scale by the numéraire \mathfrak{A}_7 , $\mathbf{p}^T \mathbf{y} = 1$.

Proof. Charging the uniform factor $(1 + r)$ on the value of all advanced inputs $\tilde{\mathbf{A}} = \mathbf{A} + \boldsymbol{\gamma} \mathbf{l}^T$ gives, commodity by commodity, (4.4). Rearranging,

$$\mathbf{p}^T \tilde{\mathbf{A}} = (1 + r)^{-1} \mathbf{p}^T,$$

so \mathbf{p}^T is a left eigenvector of $\tilde{\mathbf{A}}$ for the eigenvalue $\lambda = (1 + r)^{-1}$. Economic prices are positive, $\mathbf{p}^T > \mathbf{0}$; by Theorem 1 (III–IV) the *only* eigenvalue of the irreducible matrix $\tilde{\mathbf{A}}$ admitting a semipositive left eigenvector is $\rho(\tilde{\mathbf{A}})$. Hence $\rho(\tilde{\mathbf{A}}) = \lambda$:

$$\frac{1}{1 + r} = \rho(\tilde{\mathbf{A}}) \iff r = \frac{1}{\rho(\tilde{\mathbf{A}})} - 1,$$

which is > 0 by the surplus hypothesis $\rho(\tilde{\mathbf{A}}) < 1$: the surplus over the socio-technical inputs funds a strictly positive, uniquely determined profit rate. The same part of the theorem identifies \mathbf{p}^T with the left Perron–Frobenius eigenvector $\mathbf{q}_{\mathcal{L}}^T > \mathbf{0}$, unique up to a positive scalar. The numéraire \mathfrak{A}_7 fixes that scalar: with $\mathbf{y} = (\mathbf{I} - \mathbf{A})\mathbf{x} \geq \mathbf{0}$ and $\mathbf{p}^T > \mathbf{0}$ we have $\mathbf{p}^T \mathbf{y} > 0$, so the normalized representative $\mathbf{p}^T = \mathbf{q}_{\mathcal{L}}^T / (\mathbf{q}_{\mathcal{L}}^T \mathbf{y})$ exists and is unique. \square

Proposition 10 (Price system with surplus, wages above subsistence). *Consider a surplus economy in the sense of 26. Let $w \geq 0$, $r \geq 0$ be the ruling wage and profit rates, and write $R \equiv \frac{1}{\rho(\mathbf{A})} - 1$. Then the price system is*

$$\mathbf{p}^T = (1 + r)\mathbf{p}^T \mathbf{A} + w \mathbf{l}^T. \quad (4.6)$$

For every $r \in [0, R)$ the matrix $\mathbf{I} - (1 + r)\mathbf{A}$ is invertible with strictly positive inverse (the matrix $(1 + r)\mathbf{A}$ being irreducible) and (4.6) has the unique solution

$$\mathbf{p}^T = \mathbf{p}^T(w, r) = w \mathbf{l}^T [\mathbf{I} - (1 + r)\mathbf{A}]^{-1}. \quad (4.7)$$

For $r > R$ no semipositive price vector solves (4.6): R is the maximum admissible profit rate. The series $\mathbf{l}^T \sum_{k=0}^{\infty} (1 + r)^k \mathbf{A}^k$ converges if and only if $r < R$.

Proof. Charging $(1 + r)$ on the advanced input value $\mathbf{p}^T \mathbf{A}$ and adding the unadvanced wage term $w \mathbf{l}^T$ gives (4.6). Put $\mathbf{M} \equiv (1 + r)\mathbf{A}$, $\rho(\mathbf{M}) = (1 + r)\rho(\mathbf{A})$. By the resolvent face of Proposition 3 (II), $[\mathbf{I} - \mathbf{M}]^{-1} = \sum_{k=0}^{\infty} \mathbf{M}^k > \mathbf{O}$ exists if and only if $\rho(\mathbf{M}) < 1$, in which case it is strictly positive (\mathbf{M} irreducible), and

$$\rho(\mathbf{M}) < 1 \iff (1 + r)\rho(\mathbf{A}) < 1 \iff 1 + r < \frac{1}{\rho(\mathbf{A})} \iff r < R \equiv \frac{1}{\rho(\mathbf{A})} - 1.$$

Rearranging (4.6) as $\mathbf{p}^T [\mathbf{I} - \mathbf{M}] = w \mathbf{l}^T$ and inverting yields (4.7). *Lower-endpoint.* Evaluating equation (4.7) at $r = 0$ we get $\mathbf{p}^T = \mathbf{l}^T (\mathbf{I} - \mathbf{A})^{-1}$. *Upper-endpoint* At $r = R$, $\mathbf{M} = (1 + R)\mathbf{A}$ is irreducible with $\rho(\mathbf{M}) = 1$; let $\mathbf{q}_{\mathcal{R}} > \mathbf{0}$ be its right Perron–Frobenius eigenvector, and since $\rho(\mathbf{M}) = 1$ we have $\mathbf{M}\mathbf{q}_{\mathcal{R}} = \mathbf{q}_{\mathcal{R}}$, hence $(\mathbf{I} - \mathbf{M})\mathbf{q}_{\mathcal{R}} = [\mathbf{I} - (1 + R)\mathbf{A}]\mathbf{q}_{\mathcal{R}} = \mathbf{0}$. Right-multiplying (4.6) by $\mathbf{q}_{\mathcal{R}}$:

$$\mathbf{p}^T [\mathbf{I} - (1 + R)\mathbf{A}]\mathbf{q}_{\mathcal{R}} = w \mathbf{l}^T \mathbf{q}_{\mathcal{R}} = 0 \Rightarrow w \mathbf{l}^T \mathbf{q}_{\mathcal{R}} = 0,$$

and since $\mathbf{l}^T \mathbf{q}_{\mathcal{R}} > 0$, assuming $r = R$ forces $w = 0$. Putting $w = 0$ in (4.6) yields

$$\mathbf{p}^T = (1 + r)\mathbf{p}^T \mathbf{A} \Rightarrow \mathbf{p}^T \mathbf{A} = (1 + r)^{-1} \mathbf{p}^T,$$

so $(1 + r)^{-1} = \rho(\mathbf{A})$ (the unique eigenvalue with a positive eigenvector, Theorem 1 III–IV), i.e. $r = R$. At that corner $\mathbf{p}^T = \mathbf{q}_{\mathcal{L}}^T$ is the left-hand Perron–Frobenius eigenvector of \mathbf{A} , strictly positive. \square

Proposition 11 (Wage–profit frontier). *Consider a surplus economy as in Definition 26, which solves Proposition 10. Then the wage rate is a function of the profit rate alone,*

$$w(r) = \frac{1}{\phi(r)}, \quad \phi(r) \equiv \mathbf{I}^T [\mathbf{I} - (1+r)\mathbf{A}]^{-1} \mathbf{y} = \mathbf{I}^T \sum_{k=0}^{\infty} (1+r)^k \mathbf{A}^k \mathbf{y}, \quad (4.8)$$

defined for $r \in [0, R)$ and extended by $w(R) \equiv 0$. The map $w : [0, R] \rightarrow [0, 1]$ is a continuous, strictly decreasing bijection with $w(0) = 1$ and $w(R) = 0$. Its graph

$$\mathcal{F} = \{(r, w) : w = w(r), r \in [0, R]\}$$

is defined as the wage–profit frontier.

Proof. For $r \in [0, R)$, substitute (4.7) into \mathfrak{A}_7 : $1 = \mathbf{p}^T \mathbf{y} = w \phi(r)$. By Proposition 10 we know that $[\mathbf{I} - (1+r)\mathbf{A}]^{-1} > \mathbf{I}$, and by Definitions 26 and 18 we have that $\mathbf{y}, \mathbf{I}^T > \mathbf{0} \Rightarrow \phi(r) > 0$, hence $w(r) = 1/\phi(r) \in \mathbb{R}_{++}$. *Monotonicity and continuity.* On $[0, R)$, $\phi(r) = \mathbf{I}^T \sum_{k=0}^{\infty} (1+r)^k \mathbf{A}^k \mathbf{y}$ (Proposition 10); each coefficient $(1+r)^k$ is strictly increasing in r , the terms $\mathbf{A}^k \mathbf{y} \geq \mathbf{0}$ are not all zero, and $\mathbf{I}^T > \mathbf{0}$, so ϕ is continuous and strictly increasing. Hence, $w = 1/\phi$ is continuous and strictly decreasing, and together with the endpoint values below it is a bijection $[0, R] \rightarrow [0, 1]$. *Lower endpoint.* At $r = 0$ we have $\phi(0) = \mathbf{I}^T (\mathbf{I} - \mathbf{A})^{-1} (\mathbf{I} - \mathbf{A}) \mathbf{x} = \mathbf{I}^T \mathbf{x} = L = 1$, so $w(0) = 1$. *Upper endpoint.* By Proposition 10 we know that $r = R \iff w = 0$, hence $w(R) \equiv 0$. \square

Definition 27 (Prices as a Function of r). We can substitute (4.8) into (4.7) and obtain the following expression:

$$\mathbf{p}^T(r) = \frac{1}{\mathbf{I}^T [\mathbf{I} - (1+r)\mathbf{A}]^{-1} \mathbf{y}} \mathbf{I}^T [\mathbf{I} - (1+r)\mathbf{A}]^{-1} \quad (4.9)$$

Note that, by Propositions 10 and 11, this expression holds if and only if $r \in (0, R)$.

Definition 28 (Neumann Expansion of the Curves). Consider a surplus economy (Definition 26), with a price system as the one described in Proposition 10: where the matrix $\mathbf{M} = (1+r)\mathbf{A}$ has $\rho(\mathbf{M}) < 1$ for $r \in [0, R)$. We can write the following expressions

$$w(r) = \frac{1}{\mathbf{I}^T \mathbf{y} + \sum_{k=1}^{\infty} (1+r)^k \alpha_k}, \quad \text{with } \alpha_k = \mathbf{I}^T \mathbf{A}^k \mathbf{y}. \quad (4.10)$$

This is the Neumann expansion of $(\mathbf{I} - \mathbf{M})^{-1}$ in equation (4.8). The wage profit frontier is a power series in $(1 + r)$, and generally, nonlinear. We can use this last expression, equation (4.10), and the same expansion in equation (4.7):

$$\mathbf{p}^T(r) = \frac{\mathbf{I}^T + \sum_{k=1}^{\infty} (1+r)^k \mathbf{I}^T \mathbf{A}^k}{\mathbf{I}^T \mathbf{y} + \sum_{k=1}^{\infty} (1+r)^k \alpha_k}. \quad (4.11)$$

Each $p_j(r)$ is a ratio of two power series in $(1 + r)$, and generally, nonlinear.

Remark 3 (The three regimes of the closed system). The price vector (Proposition 10) and the wage-profit frontier (Proposition 11) split into three regimes:

$$\mathbf{p}^T(w, r) = \begin{cases} \mathbf{I}^T (\mathbf{I} - \mathbf{A})^{-1}, & r = 0, w = 1 \quad (\text{max wage}), \\ w \mathbf{I}^T [\mathbf{I} - (1+r)\mathbf{A}]^{-1}, & r \in (0, R), w \in (0, 1), \\ \mathbf{q}_{\mathcal{L}}^T \text{ with } \mathbf{q}_{\mathcal{L}}^T \mathbf{y} = 1, & r = R, w = 0 \quad (\text{max profit}); \end{cases}$$

$$w(r) = \begin{cases} 1, & r = 0, \\ \frac{1}{\mathbf{I}^T [\mathbf{I} - (1+r)\mathbf{A}]^{-1} \mathbf{y}} \in (0, 1), & r \in (0, R), \\ 0, & r = R; \end{cases}$$

4.5 A Criticism of Labour Theory of Value

Definition 29 (Direct capital and capital intensities). Consider a surplus economy (Definition 26), with a price system as the one described in Proposition 10. The *direct value of capital* is

$$\mathbf{K}^T = \mathbf{K}^T(r) = \mathbf{p}^T \mathbf{A} \in \mathbb{R}_{++}^{1 \times n}, \quad (4.12)$$

the value of the means of production used *directly* per unit of gross output of each commodity.

The (*direct*) *capital intensity* is

$$\mathbf{k}^T = \mathbf{k}^T(r) = \mathbf{K}^T \hat{\mathbf{I}}^{-1} \in \mathbb{R}_{++}^{1 \times n}, \quad k_j = \frac{K_j}{l_j} \quad (j \in \mathcal{N}), \quad (4.13)$$

the value of direct capital per unit of direct labour. We can also define:

$$K = K(r) = \mathbf{p}^T \mathbf{s} = \mathbf{p}^T \mathbf{A} \mathbf{x} = \left(\frac{1}{\mathbf{I}^T [\mathbf{I} - (1+r)\mathbf{A}]^{-1} \mathbf{y}} \mathbf{I}^T [\mathbf{I} - (1+r)\mathbf{A}]^{-1} \right) \mathbf{A} \mathbf{x} \quad (4.14)$$

$$k = k(r) = \frac{K}{L} = K, \quad \text{under } \mathfrak{A}_6. \quad (4.15)$$

As the economy-wide value of the capital stock and its capital intensity counterpart, which coincide. Following Definition 28, and substituting equation (4.11) in (4.14) we obtain:

$$K(r) = \frac{\sum_{k=0}^{\infty} (1+r)^k \mathbf{I}^T \mathbf{A}^{k+1} \mathbf{x}}{\sum_{k=0}^{\infty} (1+r)^k \alpha_k}. \quad (4.16)$$

This is ratio of two power series with strictly positive coefficients, i.e., $K(r)$ is a rational function of $(1+r)$, and, not linear in general.

Definition 30 (Vertically Integrated Coefficients). Consider a surplus economy (Definition 26), with a price system as the one described in Proposition 10. The *vertically integrated labour coefficient* vector is $\mathbf{v}^T = \mathbf{I}^T (\mathbf{I} - \mathbf{A})^{-1} \in \mathbb{R}_{++}^{1 \times n}$; and the *vertically integrated coefficient matrix* is $\mathbf{H} = \mathbf{A} (\mathbf{I} - \mathbf{A})^{-1} \in \mathbb{R}_{++}^{n \times n}$. Moreover $\rho(\mathbf{H}) = \frac{1}{R}$.

Definition 31 (Vertically Integrated Prices, Wage–Profit Frontier). Consider a surplus economy (Definition 26), with a price system as the one described in Proposition 10. Consider \mathbf{v}^T and \mathbf{H} as defined in Definition 30. Equations (4.6)–(4.7) and (4.8) can be written as:

$$\mathbf{p}^T = \mathbf{p}^T r \mathbf{H} + w \mathbf{v}^T; \quad (4.17)$$

$$\mathbf{p}^T(w, r) = w \mathbf{v}^T (\mathbf{I} - r \mathbf{H})^{-1}; \quad (4.18)$$

$$w(r) = \frac{1}{\mathbf{v}^T (\mathbf{I} - r \mathbf{H})^{-1} \mathbf{y}}. \quad (4.19)$$

$$\mathbf{p}^T(r) = \frac{1}{\mathbf{v}^T (\mathbf{I} - r \mathbf{H})^{-1} \mathbf{y}} \mathbf{v}^T (\mathbf{I} - r \mathbf{H})^{-1}; \quad (4.20)$$

This last definitions are the vertically integrated versions of the results achieved in Propositions 10 and 11. These definitions hold under the fact that $\rho(r \mathbf{H}) < 1 \Rightarrow \mathbf{I} - r \mathbf{H}$ is non-singular and has a strictly positive inverse (see Proposition 3).

Proposition 12 (Labour theory of value). Consider a surplus economy (Definition 26), with a price system as the one described in Proposition 10. Then: (I) at $r = 0$ ($w = 1$) one has $\mathbf{p}^T = \mathbf{v}^T$ for any technique; (II) for $r \in (0, R)$,

$$\mathbf{p}^T = \mathbf{v}^T \iff \mathbf{k}^T = \kappa \mathbf{e}^T \text{ for some } \kappa > 0 \iff \mathbf{1} \propto \mathbf{q}_{\mathcal{L}}.$$

Thus, uniform capital intensities are sufficient for the labour theory of value at every distribution, but become necessary only once the $r = 0$ is excluded. In general: prices of production are only independent of distribution if $r = 0$ or $\mathbf{k}^T = \kappa \mathbf{e}^T$

Proof. (I). At $r = 0$ equation (4.7) reduces to $\mathbf{p}^T = \mathbf{I}^T(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{v}^T$ (see Proposition 10), irrespective of proportions. **(II) sufficiency.** Let $\mathbf{k}^T = \kappa \mathbf{e}^T$, i.e. $\mathbf{K}^T = \kappa \mathbf{e}^T \hat{\mathbf{1}} = \kappa \mathbf{I}^T$ by (4.12)–(4.13). Substituting into (4.6), remembering that $\mathbf{p}^T \mathbf{A} = \mathbf{K}^T = \kappa \mathbf{I}^T$, we get $\mathbf{p}^T = (1+r)\kappa \mathbf{I}^T + w \mathbf{I}^T = [(1+r)\kappa + w] \mathbf{I}^T$, so $\mathbf{p}^T \propto \mathbf{I}^T$. Post-multiplying by \mathbf{A} and using $\mathbf{p}^T \mathbf{A} = \kappa \mathbf{I}^T$,

$$\kappa \mathbf{I}^T = [(1+r)\kappa + w] \mathbf{I}^T \mathbf{A} \iff \mathbf{I}^T \mathbf{A} = \frac{\kappa}{[(1+r)\kappa + w]} \mathbf{I}^T.$$

This implies that $\mathbf{I}^T > \mathbf{0}$ is a positive left eigenvector of the irreducible \mathbf{A} , by Theorem 1 (III–IV) it is the left-hand Perron–Frobenius eigenvector, $\mathbf{I}^T \propto \mathbf{q}_{\mathcal{L}}^T$, with eigenvalue $\rho(\mathbf{A}) = \frac{1}{1+R}$; thus $\frac{\kappa}{(1+r)\kappa + w} = \frac{1}{1+R} \iff (1+r)\kappa + w = (1+R)\kappa \iff w = \kappa(R-r)$. Note that $\mathbf{I}^T \mathbf{y} = \mathbf{I}^T(\mathbf{I} - \mathbf{A})\mathbf{x} = 1 - \rho(\mathbf{A}) = \frac{R}{1+R}$. And note that $(1+R)\kappa = (1+r)\kappa + w \Rightarrow \mathbf{p} = (1+R)\kappa \mathbf{I}^T$, we can post multiply this last price equation expression by \mathbf{y} (remember \mathfrak{A}_7), and obtain

$$1 = \mathbf{p}^T \mathbf{y} = (1+R)\kappa \mathbf{I}^T \mathbf{y} = (1+R)\kappa \frac{R}{(1+R)} = \kappa R \Rightarrow \kappa = \frac{1}{R} \Rightarrow w = 1 - \frac{r}{R}.$$

This is a remarkable result: the assumption of equal capital-labour ratio yields, as the general proportion in all industries $\kappa = \rho(\mathbf{H}) = \frac{1}{R}$. From here, note that since $R = \frac{1}{\rho(\mathbf{A})} - 1$, then $\rho(\mathbf{A}) = \frac{1}{1+R}$. With this expression, we can see that

$$\mathbf{p}^T = (1+R)\kappa \mathbf{I}^T = \frac{(1+R)}{R} \mathbf{I}^T = \frac{1}{1 - \rho(\mathbf{A})} \mathbf{I}^T = \mathbf{I}^T(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{v}^T.$$

Where the fourth equality uses the fact that \mathbf{I}^T is proportional to the left-hand Perron–Frobenius eigenvector. **(II), necessity.** Let $r \in (0, R)$ and suppose $\mathbf{p}^T = \mathbf{v}^T$. From $\mathbf{v}^T(\mathbf{I} - \mathbf{A}) = \mathbf{I}^T$ we have $\mathbf{v}^T \mathbf{A} = \mathbf{v}^T - \mathbf{I}^T$, so substituting $\mathbf{p}^T = \mathbf{v}^T$ into the price system (4.6) gives $\mathbf{v}^T = (1+r)\mathbf{v}^T \mathbf{A} + w \mathbf{I}^T = (1+r)(\mathbf{v}^T - \mathbf{I}^T) + w \mathbf{I}^T$. Collecting the \mathbf{v}^T terms on the left and the \mathbf{I}^T terms on the right,

$$-r \mathbf{v}^T = -(1+r-w) \mathbf{I}^T \iff \mathbf{v}^T = \frac{1+r-w}{r} \mathbf{I}^T \equiv c \mathbf{I}^T, \quad c \equiv \frac{1+r-w}{r} > 0,$$

Thus $\mathbf{v} \propto \mathbf{l}$ on a single positive scalar c , free of industry. We substitute $\mathbf{v}^T = c \mathbf{I}^T$ back into the defining identity $\mathbf{I}^T = \mathbf{v}^T(\mathbf{I} - \mathbf{A}) = c \mathbf{I}^T(\mathbf{I} - \mathbf{A})$:

$$\mathbf{I}^T = c \mathbf{I}^T - c \mathbf{I}^T \mathbf{A} \iff \mathbf{I}^T \mathbf{A} = \frac{c-1}{c} \mathbf{I}^T.$$

So $\mathbf{I}^T > \mathbf{0}$ is a positive left eigenvector of the irreducible \mathbf{A} . Hence, by Theorem 1 (III–IV), $\mathbf{I}^T \propto \mathbf{q}_{\mathcal{L}}^T$ and $\frac{c-1}{c} = \rho(\mathbf{A}) = \frac{1}{1+R} \Rightarrow c = \frac{1+R}{R}$. With $\mathbf{p}^T = \mathbf{v}^T$ the direct capital is

$\mathbf{K}^T = \mathbf{p}^T \mathbf{A} = \mathbf{v}^T \mathbf{A} = \mathbf{v}^T - \mathbf{l}^T = (c - 1) \mathbf{l}^T = \frac{1}{R} \mathbf{l}^T$, again proportional to \mathbf{l}^T on the *scalar* $c - 1 = \frac{1-w}{r} > 0$ and $w = 1 - \frac{r}{R}$. Dividing out the labour coefficients,

$$\mathbf{k}^T = \mathbf{K}^T \hat{\mathbf{l}}^{-1} = \frac{1}{R} \mathbf{l}^T \hat{\mathbf{l}}^{-1} = \frac{1}{R} \mathbf{e}^T \equiv \kappa \mathbf{e}^T, \quad \kappa = \frac{1}{R} = \rho(\mathbf{H}).$$

The capital-labour ratio is uniform all industries This closes the chain $\mathbf{p}^T = \mathbf{v}^T \Rightarrow \mathbf{l}^T \propto \mathbf{q}_{\mathcal{L}}^T \Rightarrow \mathbf{k}^T = \kappa \mathbf{e}^T$, and with sufficiency the three are equivalent on $r \in (0, R)$. Note that $K = \mathbf{K}^T \mathbf{x}$, and since we know that $\mathbf{K}^T = \frac{1}{R} \mathbf{l}^T$ its easy to see that $K = \frac{1}{R} \mathbf{l}^T \mathbf{x} = \frac{1}{R} = k$. Hence, if $r \in (0, R)$ and the labour theory of value holds or if $r \in (0, R)$ capital intensities are equal across all industries, we have:

$$\mathbf{p}^T = \mathbf{v}^T; \quad w(r) = 1 - \frac{r}{R}; \quad K = \frac{1}{R}.$$

Prices are independent of distribution and equal to the the vertically integrated labour coefficient, the wage profit frontier is linear, the capital stock is constant. \square

4.6 The Standard System

Definition 32 (Rate of Surplus). Consider a surplus economy (Definition 26), with a price system as the one characterized in Proposition 10. We define

$$\boldsymbol{\psi} = \mathbf{y} \hat{\mathbf{s}}^{-1} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\} \quad (4.21)$$

As the vector whose i -th row contains the *rate of surplus* of each industry, i.e., the net product to means of production ratio. From Definition 4.1, this implies that $\mathbf{y} = \hat{\boldsymbol{\psi}} \mathbf{s} = \hat{\boldsymbol{\psi}} \mathbf{A} \mathbf{x}$, hence:

$$\mathbf{x} = (1 + \hat{\boldsymbol{\psi}}) \mathbf{A} \mathbf{x}. \quad (4.22)$$

This last equation can be read in this sense: gross output is divided into two elements: surplus and reproduction, moreover, \mathbf{x} is the right-hand Perron–Frobenius eigenvector of matrix $\mathbf{B} = (1 + \hat{\boldsymbol{\psi}}) \mathbf{A}$ with $\rho(\mathbf{B}) = 1$.

Proposition 13 (Critical Proportion). Consider a surplus economy (Definition 26), with a price system as the one characterized in Proposition 10. Then, if the surplus ratio is uniform across all sectors, i.e., $\boldsymbol{\psi} = \psi \mathbf{e}$, we have that $\psi = R$. Where R is the critical proportion.

Proof. With $\boldsymbol{\psi} = \psi \mathbf{e}$ equation (4.22) transforms into $\mathbf{x} = (1 + \psi)\mathbf{A}\mathbf{x}$, i.e., \mathbf{x} is the right-hand Perron–Frobenius eigenvector of matrix \mathbf{A} with $\rho(\mathbf{A}) = \frac{1}{1+\psi} \Rightarrow \psi = \frac{1}{\rho} - 1$, and, by Proposition 10, this is R . A production system that has uniform net product to means of production ratios has R as this proportion. \square

Definition 33 (The Standard System). Consider a surplus economy (Definition 26), with the price system of Proposition 10. The *standard system* is the gross-output vector $\mathbf{x}_s > \mathbf{0}$ at which the surplus ratio is uniform across commodities, $\boldsymbol{\psi} = R \mathbf{e}$. By the Critical Proportion (Proposition 13), equation (4.22) reads $\mathbf{x}_s = (1 + R)\mathbf{A}\mathbf{x}_s$, equivalently $\mathbf{A}\mathbf{x}_s = \rho(\mathbf{A})\mathbf{x}_s$; hence $\mathbf{x}_s = \mathbf{q}_{\mathcal{R}}$, the right-hand Perron–Frobenius eigenvector of \mathbf{A} (Theorem 1, III), fixed in scale by $\mathbf{1}^T \mathbf{x}_s = 1$, so that the standard system employs the whole labour force of the actual economy (\mathfrak{A}_6). Its means of production are $\mathbf{s}_s \equiv \mathbf{A}\mathbf{x}_s$ and its net product—the *standard commodity*—is $\mathbf{y}_s \equiv (\mathbf{I} - \mathbf{A})\mathbf{x}_s$. The positive scalars carrying the actual industries into \mathbf{x}_s are Sraffa’s *standard multipliers*.

Proposition 14 (Standard proportions). *In the standard system (Definition 33),*

$$\mathbf{s}_s = \rho(\mathbf{A}) \mathbf{q}_{\mathcal{R}} = \frac{1}{1 + R} \mathbf{q}_{\mathcal{R}}; \quad \mathbf{y}_s = [1 - \rho(\mathbf{A})] \mathbf{q}_{\mathcal{R}} = \frac{R}{1 + R} \mathbf{q}_{\mathcal{R}},$$

so means of production and net product share the commodity composition of $\mathbf{q}_{\mathcal{R}}$ and stand in the uniform proportion $\mathbf{y}_s = R\mathbf{s}_s$. The ratio of the standard net product to the standard means of production—the standard ratio—is $R = \frac{1}{\rho(\mathbf{A})} - 1$, the maximum rate of profit of Proposition 10.

Proof. From $\mathbf{A}\mathbf{q}_{\mathcal{R}} = \rho(\mathbf{A})\mathbf{q}_{\mathcal{R}}$ we get $\mathbf{s}_s = \mathbf{A}\mathbf{x}_s = \rho(\mathbf{A})\mathbf{q}_{\mathcal{R}}$ and $\mathbf{y}_s = \mathbf{x}_s - \mathbf{s}_s = (1 - \rho(\mathbf{A}))\mathbf{q}_{\mathcal{R}}$, both proportional to $\mathbf{q}_{\mathcal{R}}$. Dividing entry by entry, $\boldsymbol{\psi} = \mathbf{y}_s \hat{\mathbf{s}}_s^{-1} = \frac{1 - \rho(\mathbf{A})}{\rho(\mathbf{A})} \mathbf{e} = \left(\frac{1}{\rho(\mathbf{A})} - 1\right)\mathbf{e} = R \mathbf{e}$ (Definition 32), which identifies the standard ratio with R . \square

Proposition 15 (The standard commodity linearises the frontier). *Adopt the standard net product as numéraire, $\mathbf{p}^T \mathbf{y}_s = 1$ (in place of \mathfrak{A}_7). Then for every $r \in [0, R]$ the wage–profit frontier is the straight line*

$$w(r) = 1 - \frac{r}{R}, \tag{4.23}$$

independent of relative prices and of the technique.

Proof. Post-multiply the price system (4.6) by $\mathbf{x}_s = \mathbf{q}_R$ and use $\mathbf{A}\mathbf{q}_R = \frac{1}{1+R}\mathbf{q}_R$ together with $\mathbf{I}^T\mathbf{q}_R = 1$:

$$\mathbf{p}^T\mathbf{q}_R = (1+r)\mathbf{p}^T\mathbf{A}\mathbf{q}_R + w\mathbf{I}^T\mathbf{q}_R = \frac{1+r}{1+R}\mathbf{p}^T\mathbf{q}_R + w \implies \mathbf{p}^T\mathbf{q}_R \frac{R-r}{1+R} = w.$$

The numéraire gives $\mathbf{p}^T\mathbf{y}_s = \frac{R}{1+R}\mathbf{p}^T\mathbf{q}_R = 1$, i.e. $\mathbf{p}^T\mathbf{q}_R = \frac{1+R}{R}$. Substituting, $w = \frac{1+R}{R} \cdot \frac{R-r}{1+R} = 1 - \frac{r}{R}$. The endpoints $w(0) = 1$ and $w(R) = 0$ agree with Proposition 11. \square

Remark 4 (An invariable measure of distribution). Against the actual net product the frontier $w(r) = 1/\phi(r)$ is in general a curved power series in $(1+r)$ (Definition 28); against the standard commodity it is exactly linear, with slope $-1/R$ a datum of the technique alone. The standard commodity is thus the measure Ricardo sought: the aggregate wage–profit trade-off it records is freed of distribution. The invariance is confined to the aggregate—individual relative prices p_i/p_j still move with r whenever capital intensities differ across industries (Proposition 12)—yet the linear w – r relation holds for every admissible technique.

4.7 Reduction to Dated Quantities of Labour

Definition 34 (Dated quantities of labour). Consider a surplus economy (Definition 26) with the price solution (4.7). For $r \in [0, R)$ the matrix $(1+r)\mathbf{A}$ has spectral radius below unity, so the Neumann expansion of $[\mathbf{I} - (1+r)\mathbf{A}]^{-1}$ (Proposition 10) applies, and

$$\mathbf{p}^T(w, r) = w\mathbf{I}^T[\mathbf{I} - (1+r)\mathbf{A}]^{-1} = w\mathbf{I}^T \sum_{k=0}^{\infty} (1+r)^k \mathbf{A}^k = \sum_{k=0}^{\infty} w(1+r)^k \mathbf{I}^T \mathbf{A}^k. \quad (4.24)$$

Define the *dated labour vector of age k* as $\mathbf{I}_k^T \equiv \mathbf{I}^T \mathbf{A}^k \in \mathbb{R}_+^{1 \times n}$, $k \in \mathbb{N} \cup \{0\}$. Its j -th entry is the labour expended k stages back in the vertically integrated process delivering one unit of commodity j : directly at $k = 0$ ($\mathbf{I}_0^T = \mathbf{I}^T$), on its means of production at $k = 1$ ($\mathbf{I}_1^T = \mathbf{I}^T \mathbf{A}$), on the means of production of those means at $k = 2$, and so on. Each layer is carried forward to the present by the profit factor $(1+r)^k$ and valued at the wage w , so the price of each commodity is an infinite sum of dated quantities of labour,

$$p_j(w, r) = \sum_{k=0}^{\infty} w(1+r)^k (\mathbf{I}^T \mathbf{A}^k)_j.$$

Remark 5 (The reduction and the non-linearity of prices). Normalising (4.24) by the numéraire \mathfrak{A}_7 ($\mathbf{p}^T \mathbf{y} = 1$) reproduces (4.11), with $\alpha_k = \mathbf{1}^T \mathbf{A}^k \mathbf{y}$ the dated labour valued against the net product. Two limits are immediate. At $r = 0$ the factors $(1 + r)^k$ collapse to unity and the reduction sums to the embodied (vertically integrated) labour, $\mathbf{p}^T = \mathbf{1}^T \sum_{k \geq 0} \mathbf{A}^k = \mathbf{1}^T (\mathbf{I} - \mathbf{A})^{-1} = \mathbf{v}^T$, recovering the labour theory of value (Proposition 12, I). For $r > 0$ each p_j is a power series in $(1 + r)$ with the non-negative, commodity-specific age profile $\{(\mathbf{1}^T \mathbf{A}^k)_j\}_{k \geq 0}$; relative prices p_i/p_j move with r precisely because these profiles differ across commodities, and the movement need not be monotone. This dispersion of dated labour is the mechanism behind the price Wicksell effect and the non-monotonic ordering of techniques examined in the next chapter.

4.8 The Integrated Wage-Commodity Sector

Adopting the Standard commodity as numéraire, Proposition 15 gave the wage–profit frontier the linear form $w = 1 - r/R$, equivalently

$$r = R(1 - w), \quad (4.25)$$

which Garegnani calls the *surplus equation*. It rests on measuring the wage as an amount of value in terms of the Standard commodity. Sraffa’s own point of departure (PCMC, §§1–8), however, treats the wage as a *bundle of commodities*, on the same footing as the fuel for the engines and the feed for the cattle. The question of this section is whether a surplus equation can be built with the wage conceived physically. The device that answers it is the *integrated wage-commodity sector*.

Notation 2. The technique is (\mathbf{A}, \mathbf{l}) of Definition 18, irreducible, with $\rho(\mathbf{A}) < 1$ and $R = \rho(\mathbf{A})^{-1} - 1$. The wage is a bundle; $\boldsymbol{\mu}$ denotes the means of production per worker of the integrated sector (distinct from the labour-input vector $\boldsymbol{\mu} = \{L_j\}$ of Definition 18); prices are normalized in *labour commanded*.

Definition 35 (Wage commodity and the labour-commanded numéraire). Let $\boldsymbol{\lambda} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ be the bundle of commodities forming one unit of the *wage commodity*, and let w be the

wage rate expressed as a quantity of wage commodity. If the economy employs total labour L , the aggregate physical wage bill is

$$\mathbf{\Lambda} \equiv L w \boldsymbol{\lambda} \in \mathbb{R}_+^n. \quad (4.26)$$

The value of one unit of wage commodity is $p_\lambda \equiv \mathbf{p}^T \boldsymbol{\lambda}$ and the wage rate in value terms is $w p_\lambda$. Following Adam Smith, values are measured in *labour commanded*, i.e. the numéraire condition is

$$w p_\lambda = w \mathbf{p}^T \boldsymbol{\lambda} = 1. \quad (4.27)$$

Proposition 16 (Prices in labour commanded). *Under (4.27) the price system $\mathbf{p}^T = (1 + r)\mathbf{p}^T \mathbf{A} + w p_\lambda \mathbf{1}^T$ reduces to $\mathbf{p}^T = (1 + r)\mathbf{p}^T \mathbf{A} + \mathbf{1}^T$, with unique solution, for $r \in [0, R)$,*

$$\mathbf{p}^T(r) = \mathbf{1}^T [\mathbf{I} - (1 + r)\mathbf{A}]^{-1} = \mathbf{1}^T \sum_{t=0}^{\infty} (1 + r)^t \mathbf{A}^t. \quad (4.28)$$

Proof. Setting $w p_\lambda = 1$ in the price equation and inverting as in Proposition 10; the Neumann series converges iff $\rho((1 + r)\mathbf{A}) < 1 \iff r < R$ (Proposition 3). This is the resolvent of Definition 34, here normalized in labour commanded: the wage is folded into the numéraire and the explicit factor w of (4.7) disappears. \square

Definition 36 (Integrated wage-commodity sector). The *integrated wage-commodity sector* (IWCS) is the integrated sector whose physical net product is the economy-wide wage bill $\mathbf{\Lambda}$. Its gross output \mathbf{x}_w , means of production \mathbf{s}_w and labour L_w are

$$\mathbf{x}_w = (\mathbf{I} - \mathbf{A})^{-1} \mathbf{\Lambda}, \quad \mathbf{s}_w = \mathbf{A} \mathbf{x}_w, \quad L_w = \mathbf{1}^T \mathbf{x}_w, \quad (4.29)$$

so that $\mathbf{x}_w = \mathbf{\Lambda} + \mathbf{A} \mathbf{x}_w$ and $\mathbf{x}_w - \mathbf{s}_w = \mathbf{\Lambda}$. Employing $L_w \leq L$ of the economy's labour, the IWCS delivers as net output exactly the wage goods of the *whole* workforce.

Proposition 17 (The profit of the IWCS is a quantity of labour). *The IWCS has two characteristic properties. (I) Net product and wages are physically the same commodity, so its profit is determinate in physical terms,*

$$\mathbf{\Pi}_w = \mathbf{\Lambda} - L_w w \boldsymbol{\lambda} = (L - L_w) w \boldsymbol{\lambda}. \quad (4.30)$$

(II) In labour commanded, the value of that profit is independent of relative prices,

$$\Pi_w = \mathbf{p}^T \mathbf{\Pi}_w = (L - L_w) w p_\lambda = L - L_w. \quad (4.31)$$

The profit per unit of labour of the sector is therefore

$$\pi_w \equiv \frac{\Pi_w}{L_w} = \frac{L - L_w}{L_w} = \frac{L}{L_w} - 1, \quad (4.32)$$

computable before any relative price is known.

Proof. (I) The workers of the IWCS receive $L_w w \boldsymbol{\lambda}$ out of the net product $\mathbf{\Lambda} = L w \boldsymbol{\lambda}$; the residual is profit. (II) Valuing with (4.27): $\mathbf{p}^T \mathbf{\Lambda} = L w p_\lambda = L$ and $\mathbf{p}^T (L_w w \boldsymbol{\lambda}) = L_w w p_\lambda = L_w$; subtract and divide by L_w . \square

Remark 6 (Marxian analogy). With L_w in the role of variable capital (labour embodied in wages) and $L - L_w$ in that of surplus-value, (4.32) is the counterpart of Marx's rate of surplus-value S/V ; the rate of profit is recovered by deflating it by a measure of capital per worker, as $r = (S/V)/(\Omega + 1)$ is recovered from the organic composition $\Omega = C/V$.

Definition 37 (Capital per worker and the profit function). Since wages are paid *post factum*, the capital advanced in the IWCS is the value of its means of production. The means of production per worker and the capital per worker are

$$\boldsymbol{\mu} \equiv \frac{1}{L_w} \mathbf{A} \mathbf{x}_w = \frac{\mathbf{s}_w}{L_w}, \quad v_w(r) \equiv \mathbf{p}^T(r) \boldsymbol{\mu} = \mathbf{1}^T [\mathbf{I} - (1 + r) \mathbf{A}]^{-1} \boldsymbol{\mu}. \quad (4.33)$$

With the rate of profit $r = \pi_w/v_w$, the *profit function* is

$$f(r) \equiv r v_w(r) = r \mathbf{1}^T [\mathbf{I} - (1 + r) \mathbf{A}]^{-1} \boldsymbol{\mu} = r \sum_{t=0}^{\infty} (1 + r)^t \mathbf{1}^T \mathbf{A}^t \boldsymbol{\mu}, \quad (4.34)$$

the profit per worker required to remunerate the capital $v_w(r)$ at the rate r .

Proposition 18 (Properties of the profit function). On $[0, R)$ the profit function (4.34) satisfies: (I) $f(0) = 0$; (II) $f'(r) > 0$; (III) $f''(r) > 0$ (convexity); (IV) if production is non-circular ($\mathbf{A}^m = \mathbf{O}$ for some $m \leq n$), f is a polynomial, finite for every finite r ; (V) if production is circular (\mathbf{A} irreducible), $f(r) \rightarrow +\infty$ as $r \rightarrow R^-$.

Proof. Write $v_w(r) = \sum_{t \geq 0} (1+r)^t c_t$ with $c_t = \mathbf{I}^T \mathbf{A}^t \boldsymbol{\mu} \geq 0$ and $c_0 = \mathbf{I}^T \boldsymbol{\mu} > 0$, convergent iff $r < R$. (I) $f(0) = 0$ is immediate. (II)–(III) As a power series in r with non-negative coefficients and a positive coefficient of degree ≥ 1 , both v_w and $f = r v_w$ have $f', f'' > 0$. (IV) If $\mathbf{A}^m = \mathbf{O}$ the sum terminates, so v_w and f are polynomials. (V) For irreducible \mathbf{A} , as $r \rightarrow R^-$ one has $\rho((1+r)\mathbf{A}) \rightarrow 1$, so $\mathbf{I}^T [\mathbf{I} - (1+r)\mathbf{A}]^{-1} \boldsymbol{\mu} \rightarrow \infty$; equivalently $w \rightarrow 0$ in (4.27) forces $p_\lambda \rightarrow \infty$, hence all prices and v_w diverge. \square

Theorem 2 (The surplus equation). *Given the technique (\mathbf{A}, \mathbf{l}) and the real wage $w\boldsymbol{\lambda}$, the rate of profit is the solution $r \in [0, R)$ of*

$$\frac{L - L_w}{L_w} = f(r), \quad (4.35)$$

the equality of the profit per worker read off the quantities (4.32) with the profit per worker required by prices (4.34).

Proof. By Proposition 17, $\pi_w = (L - L_w)/L_w$; by Definition 37, $\pi_w = r v_w(r) = f(r)$. Equate. Existence and uniqueness of the root on $[0, R)$ follow from Proposition 18: f is continuous and strictly increasing from $f(0) = 0$ to $f(R^-) = +\infty$ (circular case), so it meets the constant $\pi_w \geq 0$ exactly once. \square

Proposition 19 (Inverse wage–profit relation). *Let the value wage rate fall to $w' = w(1 - \delta)$, $0 < \delta < 1$, with $\boldsymbol{\lambda}$ and (\mathbf{A}, \mathbf{l}) unchanged. Then $\boldsymbol{\Lambda}$, \mathbf{x}_w and L_w all scale by $(1 - \delta)$, the value of the IWCS net product is again L , and the profit function f is unchanged. The left side of (4.35) rises,*

$$\pi'_w = \frac{L}{L'_w} - 1 = \frac{1}{1 - \delta} \frac{L}{L_w} - 1 > \pi_w,$$

and, f being strictly increasing, the rate of profit rises: $dr/dw < 0$.

Proof. From $\boldsymbol{\Lambda}' = Lw'\boldsymbol{\lambda} = (1 - \delta)\boldsymbol{\Lambda}$ and (4.29), $\mathbf{x}'_w = (1 - \delta)\mathbf{x}_w$ and $L'_w = (1 - \delta)L_w$. Reimposing (4.27) at w' leaves $\mathbf{p}^T \boldsymbol{\Lambda}' = L$ and $\boldsymbol{\mu} = \mathbf{s}_w/L_w = \mathbf{s}'_w/L'_w$ invariant, so v_w and f are unchanged. The displayed inequality follows from $L'_w = (1 - \delta)L_w$, and strict monotonicity of f then raises the root r of (4.35). \square

Remark 7. Geometrically, (4.35) intersects the increasing convex curve $f(r)$ with the horizontal line at height $\pi_w = (L - L_w)/L_w$; a wage cut raises the line and slides the intersection

rightwards to a higher r^* , leaving f in place. The construction thus reproduces the inverse wage–profit relation of (4.25) with the wage held as a *physical bundle* rather than as value in the Standard commodity, and through (4.28) it is the dated-labour reduction (Definition 34) read at the means of production of the wage sector.

Chapter 5

The Marginalist Theory of Distribution

5.1 Factors of Production

Production. In the **classical approach**, the production process is circular: commodities are produced by means of commodities. In the **marginalist approach**, the production process is unidirectional from factors of production to outputs. *Classes* in the classical approach we have three social classes (workers, capitalists, landowners) with three types of income (wages, profits, rents), in the marginalist approach we have three kinds of income (wage, interest and rent) and **three factors of production** (labour, capital and land): *social classes are things*, distribution variables are prices.

5.2 Households and Firms, Market Equilibrium

In this system, decisions depend on two agents (households and firms) and two markets (for commodities and for factors of production). Households sell factors of production (+receiving income, factor market) and then buy goods from the firms (-expenditure, commodity market); analogously, firms hire factors of production (-incurring costs, factor market) and sell goods to households (+revenues, commodity market).

In the marginalist theory, three items are taken as data: (i) households' preferences; (ii)

factor endowments; and (iii) the technical conditions of production. On each market, prices rise (fall) if demand > supply (demand < supply): it is assumed that *market forces* tend to push the price system towards an equilibrium. Assume $N = \{1, 2, \dots, n\}$ markets with a price vector $\mathbf{p}_t \in \mathbb{R}_{++}^n$: supply and demand are a function of prices $y_i(\mathbf{p}_t)$ and $x_i(\mathbf{p}_t) \forall i \in N$. Market dynamics for any $i \in N$ are defined as:

$$\frac{dp_i}{dt} = h[x_i(\mathbf{p}_t) - y_i(\mathbf{p}_t)], \quad h > 0,$$

For a given initial price vector $\mathbf{p}_0 \in \mathbb{R}_{++}^n$ the market dynamics determine the trajectory

$$\mathbf{p} = \mathbf{p}(t, \mathbf{p}_0).$$

An equilibrium is the price vector $\mathbf{p}^* = \mathbf{p}(t, \mathbf{p}^*) \forall t$, hence

$$\frac{dp_i}{dt} = h[x_i(\mathbf{p}^*) - y_i(\mathbf{p}^*)] = 0 \forall i \in N$$

5.3 Capital in Marginalist Theory

5.3.1 Labour-Land Model

Assumption 4 (Labour-Land Models). The following model takes the following assumptions:

- (\mathfrak{A}_1) Household preferences. (i) We know that $\exists X = \mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where X is the consumption set with a preference relationship $\succeq \subseteq X \times X$ so that $\forall x, y, z \in X$ then: $x \succeq y \vee y \succeq x$ (completeness); (ii) $x \succeq y \wedge y \succeq z \Rightarrow x \succeq z$ (transitivity); (iii) if $x > y \Rightarrow x \succ y$ (monotone preference relation); and (iv) for any sequence of pairs $\{(x^n, y^n)\}_{n=1}^{\infty}$ where $x^n \succeq y^n \forall n \in \mathbb{N}$, $x = \lim_{n \rightarrow \infty} x^n \wedge y = \lim_{n \rightarrow \infty} y^n \Rightarrow x \succeq y$. Furthermore, we can define a utility function $u : X = \mathbb{R}_+ \rightarrow \mathbb{R}$ to represent $\succeq \subseteq X \times X$ if $\forall x, y$ we have that $x \succeq y \iff u(x) \geq u(y)$, where $u'(x) > 0 \wedge u''(x) < 0$.
- (\mathfrak{A}_2) The technical conditions of production. We have a production function $f : \mathbb{R}_+^2 \rightarrow \mathbb{R}^+$, $Y = f(N, L)$ that satisfies: (i) $tY = tf(N, L) = f(tN, tL) \forall t > 0$ (constant returns to scale); (ii) $MP_L = \frac{\partial Y}{\partial L} > 0 \wedge MP_N = \frac{\partial Y}{\partial N} > 0$, $\frac{\partial^2 Y}{\partial L^2} < 0 \wedge \frac{\partial^2 Y}{\partial N^2} < 0$ (decreasing marginal productivity for L and N); and (iii) the *Inada Conditions*.

(\mathfrak{A}_3) Factor endowments. The total quantities of L and N in the economy are expressed as scalars $\bar{L} > 0$ and $\bar{N} > 0$.

(\mathfrak{A}_4) The economy consists of one commodity; its price is $p = 1$.

By assumption \mathfrak{A}_2 (constant returns to scale), we can express the production function as $y = f(n)$. Costs are $C = wL + \rho N$, and revenues are $pY = Y$. For a fixed amount of labour \bar{L} the profit maximization problem is:

$$\max_n f(n) - (w + \rho n) \Rightarrow f'(n) = \rho$$

This last result further implies that the demand for land is a function of the rent rate: $n_d = f(\rho)$ with $\frac{dn_d}{d\rho} < 0$. For the supply function we have $n_s = g(\rho)$ with $\frac{dn_s}{d\rho} > 0$. For a given endowment of labour and land (constant land-labour ratio \bar{n}), such that $\bar{n} = n_s = g(\rho)$ we get that if $\bar{n} > n \Rightarrow \downarrow \rho$ and if $\bar{n} < n \Rightarrow \uparrow \rho$: hence $\exists \rho^*$ such that $\bar{n} = n_d(\rho^*) = n_d$, i.e., there is an equilibrium rent such that supply equals demand this equilibrium rent. The same process applies analogously for a fixed amount of land. Constant returns imply, by Euler's theorem, that $Y = f(N, L) = MP_N \bar{N} + MP_L \bar{L}$, and at full employment income exhaustion gives $Y = \rho^* \bar{N} + w^* \bar{L}$, hence

$$(w^* - MP_L) \bar{L} = -(\rho^* - MP_N) \bar{N}.$$

This identity is satisfied by the factor first-order conditions $\rho^* = MP_N$ and $w^* = MP_L$, which set both sides to zero. Distribution variables are uniquely determined by the marginal productivity of factors: when the rate of remuneration of each factor equals its marginal product, under constant returns to scale, firms' profits are equal to zero.

5.3.2 Solow's One Sector Model

Consider assumption $\mathfrak{A}_1 - \mathfrak{A}_4$. Using constant returns to scale, the production function can be considered in per capita terms: $y = f(k)$. Profits are $\pi = f(k) - w - rk$, differentiating with respect to k yields $f'(k) = r$. Hence, the principle of diminishing marginal product implies: (i) $k_d = k_d(r)$; (ii) there is an inverse relationship between r and the demand for k , i.e. $\frac{dk_d}{dr} < 0$. Once r^* is determined w follows residually: $w^* = f(k^*) - r^* k^*$, with w^* being equal to the marginal product of labour.

5.3.3 Swan's Two Sector Model

The economy consists of a consumption and a capital good M : $K = pM$ and $k = pm$. The production function $Y = F(M, L)$ holds of the characteristics of \mathfrak{A}_2 , hence $y = f(m)$. Profits are $\pi = y - w - rk = y - w - rpm$; maximizing over the capital-good input m yields the first-order condition $f'(m) = rp$: capital goods m enter into the production functions; however, the interest on capital is calculated on $k = pm$. Since $p = p(r)$ and $m = m(r)$, we have that $k(r) = p(r)m(r)$, differentiating with respect to r , using the product rule, we get

$$\frac{dk}{dr} = \frac{dp}{dr}m + \frac{dm}{dr}p$$

Capital per unit of labour changes due to changes in (i) prices ($\frac{dp}{dr}m$) and (ii) quantities ($\frac{dm}{dr}p$). The former is the *price Wicksell effect*, the latter is the *real Wicksell effect*. It must be noted: there is no certainty regarding the sign of $\frac{dk}{dr}$. If $\frac{dk}{dr} > 0$ then demand for capital per unit of labour increases as r increases, which is a *spurious* result in the context of marginalist price theory: the demand of any commodity (or factor of production for that matter) has an inverse relationship with its price: this holds if and only if capital intensities are equal across industries, which is the same as to say that the two commodities are the same.

The marginal product of capital in value terms equals the interest rate if and only if $\frac{dp}{dr} = 0$. Let us verify this. From the income identity $y = w + rk = w + rpm$, differentiating with respect to r :

$$\frac{dy}{dr} = \frac{dw}{dr} + \frac{dp}{dr}rm + \frac{dm}{dr}rp + mp.$$

Differentiating $y = f(m)$ and using the first-order condition $f'(m) = rp$ yields $\frac{dy}{dr} = rp\frac{dm}{dr}$; substituting this result into the former expression and cancelling $rp\frac{dm}{dr}$,

$$\frac{dw}{dr} + k = -rm\frac{dp}{dr} \Rightarrow \frac{dw}{dr} = -k \iff \frac{dp}{dr} = 0,$$

since $r, m > 0$. The frontier slope equals minus the value of capital per worker exactly when the price Wicksell effect vanishes. The marginal product of capital *in value terms* is $\frac{dy}{dk} = \frac{dy/dr}{dk/dr}$; with $\frac{dy}{dr} = rp\frac{dm}{dr}$ and $\frac{dk}{dr} = \frac{dp}{dr}m + \frac{dm}{dr}p$,

$$\frac{dy}{dk} = \frac{rp\frac{dm}{dr}}{\frac{dp}{dr}m + \frac{dm}{dr}p} = r \iff rm\frac{dp}{dr} = 0 \iff \frac{dp}{dr} = 0.$$

The marginal productivity of capital in value terms only corresponds to the rate of profit if prices are invariant to changes in distribution, which is the same as assuming equal intensities of capital across industries, which is the same as saying that the consumption good and the capital good are the same.

5.3.4 Sraffa’s Criticism of Marginalist Theory of Capital

For Sraffa, it is not possible to order techniques of production according to their capital intensities before the fixing of interest rates. Firstly, assume two techniques (a, b) with differing capital intensities (k_a, k_b) : since prices change with income distribution, and capital is measured in value terms, the ratio $\frac{k_a}{k_b}$ varies in unpredictable ways. Secondly, if an increase in the rate of interest induces a change in technique, then, the newly adopted technique need not be less capital intensive, and does not necessarily entail a smaller output per unit of labour: the chaotic non-monotonic relationship between capital intensities and profit rates implies that, as $r \rightarrow R$, a technique might be abandoned, readopted, and abandoned again: this phenomena is called reswitching.

5.4 The Surrogate Production Function

Marginalist theory employs two notions of capital: *(i)* capital as the heterogeneous set of *capital goods* used in production, and *(ii)* capital as an *amount of value* on which interest is paid (*capitalis pars debiti*). Samuelson (1962) sought to reconcile them. His strategy was to begin with a *realistic* model — heterogeneous capital goods, fixed-coefficient methods, in order to show that it behaves *as if* capital were a homogeneous substance “jelly” J , which is governed by a differentiable production function $Y = F(J, L)$. The differentiable “parable” is meant to be a *surrogate* that reproduces the results of the realistic model.

5.4.1 The Realistic Model

There is one consumption good c and many kinds of capital goods; for each capital good there is a method producing c (using labour and that capital good) and a method producing

the capital good (using labour and itself). A method of c paired with the method of its capital good is the *technique of production*; the set of techniques is $\Theta = \{\alpha, \beta, \gamma, \dots\}$. A technique $\theta \in \Theta$ has coefficients $(a_c, a_k, \ell_c, \ell_k)$: capital-good input and labour per unit of c and per unit of the capital good. The techniques is summarized by the net output per worker y^θ , its capital goods per worker k_θ , and the value of those capital goods in terms of the consumption commodity c ,

$$j_\theta = p_\theta k_\theta, \quad (5.1)$$

which Samuelson reads as the quantity of jelly embodied in k_θ . The surrogate hypothesis is that y and j behave as if $y = f(j) = F(j, 1)$. For the parable to hold Samuelson must establish (i) $dy/dr < 0$ and (ii) $dj/dr < 0$, i.e., a higher interest rate selecting techniques with less jelly and less product per worker: $dy/dj > 0$, a putative *marginal product of jelly* with r as its price. Building the vertically-integrated sector that produces c with one unit of labour, the net output and capital per worker solve

$$\ell_c y^\theta + \ell_k k_\theta = 1, \quad a_c y^\theta + a_k k_\theta = k_\theta, \quad (5.2)$$

whence $k_\theta = \frac{a_c}{1 - a_k} y^\theta$ and

$$y^\theta = \frac{1 - a_k}{\ell_c(1 - a_k) + \ell_k a_c}, \quad k_\theta = \frac{a_c}{\ell_c(1 - a_k) + \ell_k a_c}. \quad (5.3)$$

5.4.2 The Wage–Interest Curve of a Technique

With c as numéraire, wages paid *post factum*, free competition (zero pure profit), and interest rate r , the price of the capital good and the wage solve

$$a_c(1 + r)p_\theta + \ell_c w = 1, \quad a_k(1 + r)p_\theta + \ell_k w = p_\theta, \quad (5.4)$$

giving the *wage–interest curve* of the technique and the associated capital-good price,

$$w^\theta(r) = \frac{1 - a_k(1 + r)}{\ell_c [1 - a_k(1 + r)] + \ell_k a_c (1 + r)}, \quad p_\theta(r) = \frac{\ell_k}{\ell_c [1 - a_k(1 + r)] + \ell_k a_c (1 + r)}. \quad (5.5)$$

Here $w^\theta(r)$ is not the market wage but the wage that, at a given r , yields zero profit for technique θ . The curve has four properties: (i) it is decreasing, $dw^\theta/dr < 0$; (ii) at $r = 0$ it

equals the net product per worker, $w^\theta(0) = y^\theta$ (the labour theory of value, prices proportional to embodied labour); (iii) there is a maximum rate of profit R_θ with $w^\theta(R_\theta) = 0$, namely

$$R_\theta = \frac{1 - a_k}{a_k}; \quad (5.6)$$

and (iv) writing the denominator's variable part as $(\ell_k a_c - \ell_c a_k)(1+r)$, the second derivative has the sign of $\ell_k a_c - \ell_c a_k$, so the curve is *concave* when the capital good is more capital-intensive than the consumption good ($a_k/\ell_k > a_c/\ell_c$), *convex* in the opposite case, and a straight line only when the two intensities coincide, $a_k/\ell_k = a_c/\ell_c$ — which means the two commodities are produced in the same proportions, i.e. are effectively one commodity.

5.4.3 Jelly-Capital, Wicksell Effects, and the Choice of Technique

The value of capital per worker follows from the zero-profit identity $y^\theta = w^\theta + r j_\theta$, so

$$j_\theta(r) = \frac{y^\theta - w^\theta(r)}{r} = -\frac{w^\theta(r) - y^\theta}{r - 0}, \quad (5.7)$$

i.e. j_θ is minus the slope of the chord from $(0, y^\theta)$ to (r, w^θ) : capital per worker is larger the steeper the curve. Along a single technique a *convex* curve gives $dj/dr < 0$ (well-behaved), but a *concave* curve gives $dj/dr > 0$: the value of capital per worker *rises* with the interest rate. This perverse movement is the **price Wicksell effect**, and it already breaks requirement (ii) of the parable.

Facing the menu Θ at a given (r, w) , firms choose the technique maximizing profit $\pi = y^\theta - r j_\theta(r) - w$. Since the wage enters every technique identically, the comparison — and hence the choice — is independent of w , and the problem reduces to $\max_\theta [y^\theta - r j_\theta(r)] = \max_\theta w^\theta(r)$. **The technique in use at the rate r is the one paying the highest wage**, i.e. the technique on the outer envelope of the wage–interest curves,

$$w(r) \equiv \sup_{\theta \in \Theta} w^\theta(r), \quad \theta(r) \equiv \{\theta \in \Theta : w^\theta(r) = w(r)\}. \quad (5.8)$$

Switch points (where two curves cross) aside, $\theta(r)$ is single-valued, so one may set $y(r) \equiv y^{\theta(r)}$ and $j(r) \equiv [y(r) - w(r)]/r$. A *surrogate production function* is a differentiable f with $y(r) = f(j(r))$, $f'(j) > 0$, $f''(j) < 0$.

Samuelson’s construction rests on the assumption that the wage–interest curves are **linear**. If they are, capital-good prices are independent of r , each technique satisfies $j_\theta = -dw^\theta/dr$ (a constant), techniques order unambiguously by jelly content, and both y and j fall as r rises. Differentiating the envelope identity $y(r) = r j(r) + w(r)$ and using $j = -dw/dr$,

$$\frac{dy}{dr} = j + r \frac{dj}{dr} + \frac{dw}{dr} = r \frac{dj}{dr} \implies \frac{dy}{dj} = r, \quad (5.9)$$

which *looks like* an equality between the marginal product of jelly and the rate of interest. **But (5.9) depends entirely on the linearity assumption**, and linearity holds only when consumption and capital goods have equal capital intensity — the one-commodity case (Solow), where the surrogate is superfluous because the real production function already exists.

5.4.4 Reswitching, Capital Reversing, and Conclusions

Drop linearity and capital-good prices vary with r . With $\Theta = \{\alpha, \beta\}$, technique α is used where $w^\alpha(r) > w^\beta(r)$ and β where $w^\alpha(r) < w^\beta(r)$; a switch point solves $w^\alpha(r) = w^\beta(r)$. By (5.5) this is a *second-degree* equation in r , hence admits *up to two* solutions — not the single switch of the linear case. If the capital good is more capital-intensive than the consumption good under α (w^α convex) and less so under β (w^β concave), there are two switch points $r' < r''$ and **reswitching** occurs: α is optimal on $[0, r']$ and on $[r'', R_\alpha]$, but not on (r', r'') — the same technique is abandoned and later re-adopted as r rises. Consequently a rise in r may produce $\Delta j/\Delta r > 0$ (through the price effect along w^β and a real jump in jelly per worker at r'') and $\Delta y/\Delta r > 0$ (a jump in net product per worker from y^β to y^α at r'') — **capital reversing**, contradicting both $dj/dr < 0$ and $dy/dr < 0$.

Samuelson’s attempt therefore fails: the surrogate production function is assured only with a single kind of capital good (Solow’s one-commodity model), where it is redundant. A jelly-capital that can assume the shape of every capital good does not exist. Three conclusions follow.

- (i) **Capital is not a factor of production** on the same footing as labour. In general there is no unambiguous way to call one method more “capital-intensive,” “mech-

anized,” or “roundabout” than another, except in the ex-post tautological sense of being adopted at a lower interest rate.

- (ii) **The neoclassical parable does not hold.** A fall in the rate of interest need not raise either capital per worker or output per worker; the tale of an ever more “roundabout” and “productive” technique as r falls is not universally valid.
- (iii) **The rate of interest is not the price of capital.** Were it so, the quantity of capital would be the wealth on which the interest yield is reckoned; the value of capital being itself a function of r , this circularity is precisely what breaks down.

Chapter 6

Keynesian Economics

6.1 Macroeconomic Equilibrium: Two Closures

In a closed household–firm system, income is split between consumption and saving and expenditure between consumption and investment, so that the equilibrium of income and expenditure is the equality of saving and investment:

$$Y = C + S, \quad E = C + I, \quad Y = E \iff S = I, \quad (6.1)$$

with $S = Y - C(Y) \equiv S(Y)$ and $I = I(i)$. This equality can be achieved through two opposing closures. In the **neoclassical** closure, flexibility of the real wage secures full employment of the labour force \bar{N} , fixing income at $Y = P \phi(\bar{N})$; the interest rate then equilibrates full-employment saving with investment, $S(\bar{Y}) = I(i^*)$: saving determines investment through movements in the interest rate i . In the **Keynesian** closure, the interest rate equilibrates liquidity preference with the money supply, $L(i^*) = \bar{M}$; the investment thereby determined, $I = I(\bar{i})$, is matched by saving through a movement of *income*, $S(Y) = I$, and employment follows residually, $N^* = \phi^{-1}(Y^*) \leq \bar{N}$. **The decisive conclusion is that macroeconomic equilibrium is compatible with involuntary unemployment.**

6.2 Investment and the Marginal Efficiency of Capital

For a given capital-asset with expected prospective yields Q_1, \dots, Q_n and supply (replacement) price C_R , the **marginal efficiency of capital** mek is the discount rate equating the present value of the yields to the supply price,

$$C_R = \sum_{t=1}^n \frac{Q_t}{(1 + mek)^t}. \quad (6.2)$$

The yields Q_t reflect entrepreneurs' long-term expectations — the “animal spirits.” Aggregating the asset schedules, investment is pushed to the point where the marginal efficiency of capital in general equals the market rate of interest, $mek = i$, which yields an inverse relation between investment and the interest rate *without recourse to the marginalist theory of capital*. The argument has, however, a weak point: Q_t and C_R are treated as independent of distribution, whereas a change in i alters relative prices and hence shifts the whole schedule in an unpredictable way. Keynes implicitly assumes fixed prices — an assumption to which the capital-theory debate returns below.

6.3 The Consumption Function and the Multiplier

Keynes takes consumption to depend mainly on income, increasing in it but less than proportionally, with a falling average propensity:

$$C = C_0 + cY, \quad C_0 > 0, \quad 0 < c < 1, \quad (6.3)$$

so that $S = (1 - c)Y - C_0$, the marginal propensity to consume is c , to save $1 - c$, and the average propensity $C/Y = C_0/Y + c$ declines. Imposing the equilibrium $E = C + I = Y$ and iterating the adjustment of output to expenditure generates a geometric series whose sum is

$$Y^* = \frac{1}{1 - c} (C_0 + I_0), \quad (6.4)$$

with **multiplier** $1/(1 - c) > 1$. At Y^* one has $S(Y^*) = I$ for *any* interest rate: it is investment that determines the corresponding saving, not the reverse. Once Y^* is fixed, employment is determined by $N^* = \phi^{-1}(Y^*)$, in general below \bar{N} , leaving room for an increase in aggregate demand without pressure on money prices.

6.4 Liquidity Preference and the Interest Rate

For Keynes the interest rate is a monetary phenomenon: it equilibrates the demand for money with its supply, not the demand for and supply of new capital. The demand for money has a transactions-plus-precautionary component proportional to income and a speculative component inverse in the interest rate,

$$L = L_1(Y) + L_2(i), \quad i = \frac{\text{coupon}}{\text{security price}}, \quad (6.5)$$

the speculative motive turning on the gap between the current rate i and the expected rate i^{exp} (when $i \leq i^{exp}$ agents prefer money to securities). With the money stock \bar{M} set by the central bank, equilibrium on the money market is

$$\bar{M} = L_1(Y) + L_2(i). \quad (6.6)$$

6.5 Keynes' Effect and the Flexibility of Money Wages

Keynes retains the marginal-productivity condition $W/P = \phi'(N)$ but reverses its reading: the level of employment N determines the real wage W/P , not conversely. Should workers, faced with $N^* < \bar{N}$, accept a cut in money wages? The mechanism by which such a cut might raise employment is **Keynes' effect**,

$$N^* < \bar{N} \Rightarrow W \downarrow \Rightarrow P \downarrow \Rightarrow L_1 \downarrow \Rightarrow i \downarrow \Rightarrow I \uparrow \Rightarrow Y \uparrow \Rightarrow N \uparrow. \quad (6.7)$$

Keynes holds that the chain can break for two reasons: (i) investment depends not only on i but on W , since a wage cut may depress entrepreneurial expectations, so the net effect on I is negligible; and (ii) the **liquidity trap**, where at very low interest rates an increase in the demand for securities is met by an equal increase in supply, leaving i unmoved. **One cannot therefore conclude that $W \downarrow$ raises N ; workers' resistance to money-wage cuts is perfectly rational.** Moreover, real-wage reductions in practice occur through a rise in P rather than a fall in W , and money-wage cuts strike sectors unevenly, so labour resists the change in the structure of relative wages.

6.6 The Neoclassical Synthesis

Hicks (1937) and Modigliani (1944) sought to reabsorb Keynes into the mainstream. Their verdict is that the consistency of equilibrium with involuntary unemployment is due *entirely* to the assumption of rigid wages, and not to liquidity preference — except in the limiting liquidity-trap case. The natural rate of interest r^* is the level equilibrating the given supply of capital \bar{K} with its demand $K(r)$; with the nominal rate $i = r + \dot{P}$, the central bank follows a Wicksellian rule, raising i_{CB} when $\dot{P} > 0$ and lowering it when $\dot{P} < 0$, so that $\dot{P} = 0$ if and only if $i = r^*$. **In this reading the Keynesian outcome is a special case — a disequilibrium sustained by nominal rigidity — within the general marginalist equilibrium.**

6.7 The Capital-Theory Debate and Aggregate Demand

The synthesis stands or falls with the investment schedule $I(r)$, and here the debate on capital theory is decisive. We know that, in general, the demand for capital $K = K(r)$ is *not* a monotonically decreasing function of r ; hence neither is investment $I = I(r)$. It follows that even if $W \downarrow \Rightarrow r \downarrow$, there is no assurance that $I \uparrow$, and more generally no certainty that a fall in the interest rate raises aggregate demand. **The guarantee on which both Keynes' effect and the neoclassical synthesis rest — a downward-sloping $I(r)$ — is withdrawn; the inverse relation between the interest rate and aggregate demand is not a theorem.** Involuntary unemployment thus survives as a substantive possibility, independently of wage rigidity.

6.8 The Cambridge Equation

Investment plays a double role. In the short run it raises income through the multiplier, $Y = \frac{1}{1-c}(C_0 + I)$; in the long run it accumulates capital and so sets the growth rate, $g = \Delta Y/Y = \Delta K/K = I/K$. Under the classical saving hypothesis that wages are entirely consumed ($s_w = 0$), saving is $S = s_c \Pi$, and the saving–investment equilibrium $I = s_c \Pi$ gives

the **Cambridge equation**

$$g = \frac{I}{K} = \frac{s_c \Pi}{K} = s_c r. \quad (6.8)$$

With investment governed by animal spirits, \bar{I} given and K given fix the actual growth rate $\bar{g} = \bar{I}/K$; the propensity to save then yields the profit rate $\bar{r} = \bar{g}/s_c$, and once \bar{r} is known the Sraffian price equations determine w and relative prices. The equilibrium of saving and investment is thus achieved by an adjustment in *income distribution* rather than in output: in $I = s_c r K$, the data s_c and K and the independent variable I leave r as the endogenous variable.

6.9 Garegnani's Criticism of the Cambridge Equation

Garegnani distinguishes two notions of profit rate: the **normal** rate r^* appearing in Sraffa's price equations — where distribution is at its normal level, quantities match effectual demand, and the dominant technique is used at the normal degree of capacity utilization — and the **actual** rate \bar{r} . The post-Keynesian authors (Kaldor, Robinson, Pasinetti) conflate two distinct forms of (6.8),

$$s_c \bar{r} = \bar{g} \quad (\text{actual}), \quad s_c r^* = g^* \quad (\text{normal}). \quad (6.9)$$

In the first, \bar{g} is exogenous and \bar{r} the unknown — but \bar{r} cannot be the rate of Sraffa's equations, since arbitrary investment decisions do not bring about the normal degree of utilization of durable capital. In the second, g^* is not determined independently of r^* by actual investment. **The Cambridge equation cannot at once close distribution and feed Sraffian normal prices unless the normal/actual distinction is respected:** the normal degree of utilization is the one entrepreneurs desire and base investment upon, generally below the technical maximum, and need not coincide with the realized average.

Chapter 7

International Trade

The chapter follows the fortunes of the comparative-advantage doctrine as the classical assumption of national capital immobility is progressively relaxed. The lesson is that, once capital is free to move, it is *absolute* and not comparative advantage that governs the pattern of trade, and the mutual benefit of trade is no longer assured.

7.1 Ricardo's Theory of Trade

For Ricardo, international trade benefits all countries involved, the benefit residing in the quantity of product obtainable with a given employment of labour. It rests on *comparative*, not absolute, advantage: a country may gain from trade even if its partner produces every commodity with less labour. The classic illustration has England and Portugal, cloth and wine, with labour requirements

$$\text{England: } 100 L \rightarrow X_c, 120 L \rightarrow X_w; \quad \text{Portugal: } 90 L \rightarrow X_c, 80 L \rightarrow X_w.$$

The comparative-advantage condition is the inequality of relative labour costs,

$$\frac{80}{90} < \frac{120}{100} \iff \frac{\ell_w^P}{\ell_c^P} < \frac{\ell_w^E}{\ell_c^E}, \quad (7.1)$$

so Portugal has the comparative advantage in wine and England in cloth, even though Portugal holds the absolute advantage in both ($80 < 120$ and $90 < 100$). The gain is mutual: exchanging wine embodying 80 years of labour, Portugal obtains cloth that would

have cost her 90; giving cloth embodying 100, England obtains wine that would have cost it 120. **The doctrine turns essentially on the immobility of capital across countries** — the natural disinclination of capital to emigrate — for an exchange of the produce of 100 Englishmen for that of 80 Portuguese could never occur between individuals of the same country.

7.2 Absolute and Comparative Advantage in a Ricardian Model

Let there be two countries $J = A, B$ and two commodities produced by unassisted labour, with ℓ_i^J the labour to produce one unit of commodity i in country J , and a common wage-commodity bundle (λ_1, λ_2) serving as numéraire. Country A has a **comparative** advantage in commodity 1, and B in commodity 2, when

$$\frac{\ell_1^A}{\ell_2^A} < \frac{\ell_1^B}{\ell_2^B}, \quad (7.2)$$

and an **absolute** advantage in commodity 1 when $\ell_1^A w^A < \ell_1^B w^B$; absolute advantage in each commodity implies comparative advantage in it. Three cases follow.

Case 1 (wages fixed, capital immobile — Ricardo). The pattern of trade is set by comparative advantage; with A producing 1 and B producing 2, the prices and profit rates satisfy

$$\ell_1^A w^A (1 + r^A) = p_1, \quad \ell_2^B w^B (1 + r^B) = p_2, \quad \lambda_1 p_1 + \lambda_2 p_2 = 1, \quad (7.3)$$

a system with one degree of freedom, leaving room for incomplete specialization.

Case 2 (wages fixed, capital mobile). The profit rate must be uniform across countries, $\max\{r_1^A, r_1^B\} = \max\{r_2^A, r_2^B\}$. Two outcomes are possible: either A has the absolute advantage in 1 and B in 2, or A has the absolute advantage in *both*, in which case B is driven out — **“desertification.”**

Case 3 (wages flexible, capital mobile). With a uniform profit rate, wages must adjust so that no country holds an absolute advantage in both commodities; assuming $\ell_1^A/\ell_2^A < \ell_1^B/\ell_2^B$,

$$\frac{\ell_1^A}{\ell_1^B} \leq \frac{w^B}{w^A} \leq \frac{\ell_2^A}{\ell_2^B}, \quad (7.4)$$

with one of the two inequalities holding as an equality. **Once capital is mobile, the wage gap and absolute advantage — not comparative advantage alone — govern the international division of labour.**

7.3 Sraffa’s Theory of Value and International Trade

Now let the two commodities be *basic*, produced by labour and by themselves, with capital mobile and labour immobile, and physical wages given in each country with country *A* the high-wage one, $w_i^A \geq w_i^B$. Wages entering as advanced inputs define the socio-technical coefficients

$$\tilde{a}_{ih}^J = a_{ih}^J + l_i^J w_h^J, \quad i, h = 1, 2, \quad J = A, B. \quad (7.5)$$

Writing p for the price of commodity 1 in terms of commodity 2 and imposing a uniform profit rate, each of the four production methods yields a relation between p and r ,

$$(1 + r)(\tilde{a}_{11}^A p + \tilde{a}_{12}^A) = p, \quad (1 + r)(\tilde{a}_{21}^A p + \tilde{a}_{22}^A) = 1, \quad (7.6)$$

$$(1 + r)(\tilde{a}_{11}^B p + \tilde{a}_{12}^B) = p, \quad (1 + r)(\tilde{a}_{21}^B p + \tilde{a}_{22}^B) = 1, \quad (7.7)$$

the methods producing commodity 1 giving a direct p – r relation and those producing commodity 2 an inverse one. Competition with mobile capital selects, at each price, the method yielding the highest profit rate, so the operated technique is the outer envelope of these curves; their intersection fixes the international (p, r) .

Two readings follow. Under **unequal exchange** (Emmanuel), *A* is the high-wage “centre” and *B* the low-wage “periphery,” and trade allows *A* to intercept part of the surplus-value produced in *B* by virtue of its lower wages. Under **exclusion from trade** (Parrinello), it may happen that the maximum profit rate is reached with *both* commodities produced in the same country; with capital mobile, no capitalist would then invest in the other, and trade is not mutually beneficial. **The conclusions are:** (i) when the pattern of trade is endogenous and capital mobile, absolute advantage is what matters; (ii) if one country has the absolute advantage in both sectors, capital concentrates there; (iii) an excluded country has only such recourses as restricting capital mobility, the emigration of its workers, or a fall in its wages — which may itself be

ineffective.

7.4 Appendix: Ricardo on the Trade Balance and Money Prices

The comparative-cost theory is stated in relative prices, yet the decisions to buy at home or abroad turn on money prices and the exchange rate. With gold the common numéraire, the money prices of wine and cloth in each country are the labour requirements valued at the gold wage and marked up for profit,

$$P_w^P = 80 W_P(G_P)(1 + r_P), \quad P_c^P = 90 W_P(G_P)(1 + r_P), \quad (7.8)$$

$$P_w^E = 120 W_E(G_E)(1 + r_E), \quad P_c^E = 100 W_E(G_E)(1 + r_E), \quad (7.9)$$

where the gold wage $W_J(G_J)$ rises with the country's gold stock (a larger stock lowers the purchasing power of gold). If at some arbitrary distribution a country runs a deficit, gold flows out, its gold wage and money prices fall, and the balance is restored, until a distribution (G_P^*, G_E^*) equalizes the money value of imports and exports. **This is Hume's specie-flow mechanism: monetary adjustment reconciles money prices with the barter pattern dictated by comparative advantage.**

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