Spectral Structure and the Near-Linearity of Wage–Profit Curves: Evidence from U.S. Input–Output Accounts

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Abstract

Empirical wage–profit curves in market economies are strikingly near-linear, this is a a robust stylized fact at odds with theoretically admissible non-linear shapes related to the capital controversies of the 1960s–70s. A comprehensive explanation remains elusive. We characterize the theoretical and empirical productive-structure features underlying near-linear wage curves within a general family of linear production models. Evaluating these restrictions using U.S. Benchmark Input–Output Accounts (1977–2017, 400+ sectors), we identify three robust structural regularities: (*i*) close alignment between the vertically integrated labour coefficient vector $\mathbf{v}^T$ and the left-hand Perron–Frobenius eigenvector $\mathbf{q}_{\mathcal{L}}$; (*ii*) strong alignment between the net output $\mathbf{y}$ and the right-hand Perron–Frobenius eigenvector $\mathbf{q}_{\mathcal{R}}$; and (*iii*) a clustering of the subdominant spectrum of $\mathbf{A}$, of the eigenlabours $\boldsymbol{\alpha}^T = \mathbf{v}^T\mathbf{Q}$ and the eigenoutputs $\boldsymbol{\chi} = \mathbf{Q}^{-1}\mathbf{y}$ around zero (under a fixed biorthogonal normalisation of $\mathbf{Q}$). These regularities are stable across time, contributing to debates on capital theory and linear production models.

Keywords: Wage–profit curve; capital theory; spectral representation; U.S. Benchmark Input–Output Accounts

Contents

1 Introduction

The computation of linear production models using observed input-output accounts has produce many stylized facts in the relationship between relative prices and distribution. One of these regularities is the near linearity of the wage rate-profit rate curve, or wage curve. Across different countries, time periods, and levels of industrial aggregation, as well as within various linear production models, including different numéraire vectors, the empirical wage curves lack of multiple changes in direction and are well approximated by linear functions.[1]

One relevance of these stylized facts lies in the acute contrast they maintain with the complex shapes—characterized by multiple inflections and "wiggles" —that are theoretically admissible in the Sraffa/Leontief theoretical linear production models. During the Cambridge Capital Controversies of the 1960s and 1970s the logical possibility of capital paradoxes such as reswitching and reverse capital where rooted at the nonlinearity of the wage curves. In addition, both theoretical and empirical linear production models, although differing in their parameters, they

[1] Other regularities are the near linearities of industry-level variables such as relative prices, capital intensities, and capital-output ratios.

have the same mathematical structure, so it is still paradoxical why empirical wage curves are nearly linear when it depends on a polynomial the order of which depends on the number of industries in the model.

The contrast between the potential complexity allowed by the theoretical model and the observed simplicity of empirical curves has created a significant paradox in the literature. It has sparked a growing interest among researchers in identifying the specific characteristics of the productive structure —namely the interaction between the input matrix, the labour vector, and the numéraire vector— that are responsible for such robust stylized facts.

Since the era of the capital controversies, theory has identified two conditions under which the linearity of the wage curve is guaranteed.[2] First, when the labour vector is proportional to the 'price' Perron-Frobenius (P-F) eigenvector of the input matrix. In such case, industries possess uniform capital intensities and there is no price Wicksell effect. Second, when the output numeraire vector is proportional to the 'quantity' P-F eigenvector of the input matrix, that is, to Sraffa's standard commodity. In the latter case, the rates of physical surplus of each commodity is uniform. While these conditions provide clear mathematical solutions, they were traditionally viewed as highly restrictive (irregular, a fluke) special cases (Schefold, 1976).

Since the work of Schefold (2008, 2013) and Mariolis and Tsoulfidis (2011, 2016), the study of the stylized facts in empirical linear production models has been conducted through the spectral representation of the curves. This approach consist of changing the coordinate system of input matrix and the labour and numeraire vectors into that defined by the eigenvectors of the input matrix. This change of coordinates allows to identify the conditions for linearity from constraints in three types of parameters (Schefold, 2023; Torres-González, 2022): (i) the eigenvalues of the input matrix, (ii) the eigenlabours or the j-spectral labour component, or the projection of the labour vector onto the spectral subspace associated with each eigenvalue, and (iii) the eigenoutputs or the i-spectral output component, the projection of the numéraire vector onto the spectral subspace associated with each eigenvalue. Under this representation the two cases for linearity consist of all the eigenlabors and eigenoutputs equal to zero except for the the one associated to the P-F eigenvalue.

Currently, there are two hypotheses, or more precisely, two sets of constraints, on the characteristics of the productive structure to explain the near linearity of the curves —One of these is a particular case of the other. The common element in both is that they depend on sufficiently small subdominant eigenvalues, a condition that depends solely on the characteristics of the input matrix. Schefold (2008, 2013) shows that in the theoretical models, in the extreme case of

[2] Moreover, some consider that these are the only considerations Petri (2021, p. 117-9). cf. Miyao (1977).

zero subdominant eigenvalues the wage curve is a hyperbola. To reach the linearity of the curve Schefold proposes two additional constraints. First, a zero covariance between two vectors that represent deviations between two vectors: the labour vector and the 'price' P-F eigenvector and the numeraire vector and the 'quantity' P-F eigenvector. Finally, he assumes that the simple average of the deviations between labour vector and the 'price' P-F- eigenvector are zero. The second hypothesis, postulated by Mariolis and Tsoulfidis (2011, 2016) consist only on Schefold's first constraint: sufficiently small subdominant eigenvalues.

Within the literature, there has been numerous studies that show that the subdominant eigenvalues of different input matrices cluster around zero —they are not distributed uniformly in the spectral circle. The rank plot of the eigenvalues moduli show an exponential-type of decay, showing that most of the eigenvalues have a small moduli.[3] These characteristics of the eigenvalues have been mentioned to support the two hypotheses just mentioned.

However, there is a fundamental problem with these two hypothesis: in spite of this tendency to cluster around zero, there is a significant quantity of subdominant eigenvalues with an important magnitude (Ferrer-Hernández & Torres-González, 2022, pp. 37-8). These non-zero eigenvalues imply that the high order coefficients of the polynomial determining the wage curve are different from zero, with the potential to generate multiple wiggles, unless other constraints are imposed to these coefficients.

As we shall see in Section 2, zero subdominant eigenvalues are a sufficient condition for a hyperbolic wage curves, not a necessary condition. Other parameters, like the eigenlabours or the eigenoutputs can contribute to generate linear, hyperbolic or nearly linear wage curves.[4]

Ferrer-Hernández and Torres-González (2022) and Torres-González (2022) has shown that the multiplication of the subdominant eigenlabors and eigenvalues cluster around zero with a much faster rate of decay, having a smaller quantity of elements with a nonzero value associated with a smaller magnitude. These results where shown to explain the near linearity of the price curves, but they also constribute to explaining the near linearity of the wage curve. In these patterns it has shown that the eigenlabors contribute significantly to these constraints in the productive structure, hence the authors conclusion that the near linearity in prices curves can be explained by the combined effect of the parameters, that is, by the characteristics of the input matrix and its relationship with the labour vector.

Currently, there hasn't ben reported the empirical evidence of the eigenoutputs which, together with the eigenvalues and eigenlabors can explain the behavior of the wage curves. This

[3] See, Mariolis and Tsoulfidis (2016, Ch. 5 and 6), Torres-González and Yang (2019), and Shaikh et al. (2023).

[4] See Torres-González (2022, p. 636) and Schefold (2023, Appendix).

paper contributes to this literature by providing a theoretical and empirical characterization of the productive-structure features underlying nearly linear wage–profit curves. First, it derives the conditions for linear, hyperbolic, and approximately linear wage curves within a general family of linear production models. These conditions identify parameter restrictions capable of constraining the curvature of the theoretical wage–profit schedule.

Second, the paper evaluates these restrictions using the U.S. Benchmark Input–Output Accounts for 1997–2017, with more than 400 sectors. The results show that the near linearity of the empirical wage curve is driven by three robust structural regularities: a close alignment between the labor vector and the left Perron–Frobenius eigenvector of the input matrix; a strong alignment between alternative numéraire vectors and the right Perron–Frobenius eigenvector; and a clustering of subdominant eigenvalues near zero. These regularities are stable across years, suggesting the existence of stylized facts in the productive structure of the U.S. economy.

2 A model of production with one technique

2.1 The Sraffian Model of Prices of Production

Consider the Sraffian price model of an economy producing $n \in \mathbb{N}$ commodities by means of commodities with no joint production (each industry produces exactly one commodity), with only basic commodities (each commodity participates directly or indirectly in the production of all commodities), non-durable means of production (only circulating capital, no fixed capital), one given technique of production and social output, and long-period reproducible positions.[5] The model is presented by the following system (1)–(6):[6]

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{y} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} \tag{1}$$

$$\mathbf{s} = \mathbf{A}\mathbf{x} \tag{2}$$

$$\mathbf{p}^T = (1+r)\mathbf{p}^T\mathbf{A} + w\mathbf{l}^T = w\mathbf{l}^T[\mathbf{I} - (1+r)\mathbf{A}]^{-1} \tag{3}$$

$$Y = \mathbf{p}^T\mathbf{y} = 1 \tag{4}$$

$$L = \mathbf{l}^T\mathbf{x} = 1 \tag{5}$$

$$r = r^* \in \mathbb{R}_+ \tag{6}$$

The quantities system (1)-(2) indicates that the production of the n divisible commodities $\mathbf{x} \in \mathbb{R}^n_{++}$ uses the stock of produced commodities $\mathbf{s} \in \mathbb{R}^n_+$ and a quantity of homogeneous labour according to the technique of production $\mathcal{T} = \{(\mathbf{A}, \mathbf{l}); \mathbf{x}\}$, where $\mathbf{A} \in \mathbb{R}^{n\times n}_+$ and $\mathbf{l} \in \mathbb{R}^n_{++}$ are the input-coefficients matrix and the labour-coefficients (column) vector, respectively. The technique $\mathcal{T}$ summarises the interindustry structure of technology, $(\mathbf{A}, \mathbf{l})$, together with the social output, $\mathbf{x}$. $\mathcal{T}$ is characterised by the following assumptions:

Assumption 1 (Technique $\mathcal{T}$)**.** *The technique* $\mathcal{T} = \{(\mathbf{A}, \mathbf{l}); \mathbf{x}\}$ *satisfies:*

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[5] The basics of the model are outlined in Pasinetti (1977). For an exposition of long-period positions see Garegnani (1976), Kurz and Salvadori (1995), Petri (2004) and Foley (2011, 2016).

[6] This paper uses the following notation: $N = \{1, 2, \dots, n\}$, where N is the set of indices for industries and commodities. Bold-faced letters represent vectors (lower-case) and matrices (upper-case), whereas scalars are represented by regular characters. All vectors are column vectors unless specified otherwise; the transpose is denoted by the superscript T. The symbols a_i and b_j denote the i-th and j-th coordinates of column vectors $\mathbf{a}$ and $\mathbf{b}$, respectively. For vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$: $\mathbf{a} \geqq \mathbf{b}$ means $a_i \geq b_i$ $\forall i \in N$; $\mathbf{a} \geq \mathbf{b}$ means $\mathbf{a} \geqq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$; and $\mathbf{a} > \mathbf{b}$ means $a_i > b_i$ $\forall i \in N$. We write $\mathbb{R}^n_+ \equiv \{\mathbf{a} \in \mathbb{R}^n : \mathbf{a} \geqq \mathbf{0}\}$ and $\mathbb{R}^n_{++} \equiv \{\mathbf{a} \in \mathbb{R}^n : \mathbf{a} > \mathbf{0}\}$, with analogous notation for matrices ($\mathbb{R}^{n\times n}_+$, $\mathbb{R}^{n\times n}_{++}$) and for row vectors ($\mathbb{R}^{1\times n}_+$, $\mathbb{R}^{1\times n}_{++}$). Let $\mathbf{M} \in \mathbb{R}^{n\times n}$ be any $n \times n$ matrix, with m_{ij} its (i, j) entry for $i, j \in N$. $\mathbf{M}_{(i)}$ and $\mathbf{M}^{(j)}$ denote the i-th row and j-th column of $\mathbf{M}$, respectively. $\hat{\mathbf{a}} \equiv \text{diag}\{a_i\}$ denotes the diagonal matrix formed from vector $\mathbf{a}$. The summation vector is denoted $\mathbf{e} \equiv (1, 1, \dots, 1)^T$. The null vector and null matrix are denoted $\mathbf{0}$ and $\mathbf{O}$, respectively, and $\mathbf{I} \equiv \text{diag}\{1, 1, \dots, 1\}$ is the identity matrix.

($\mathfrak{A}_1$) *Labour is indispensable,* $\mathbf{l} > \mathbf{0}$*, and* $\mathbf{A} \geqq \mathbf{O}$*.*

($\mathfrak{A}_2$) $\mathbf{A}$ *is primitive (hence irreducible) and diagonalizable.*

($\mathfrak{A}_3$) *The system is viable:* $\lambda_{\mathbf{A},1} < 1$*.*

Under $\mathfrak{A}_1$-$\mathfrak{A}_2$, $\lambda_{\mathbf{A},1}$ is the simple, strictly dominant Perron–Frobenius (P-F) eigenvalue of matrix $\mathbf{A}$, associated with the left- and right-hand P-F eigenvectors $\mathbf{q}_{\mathcal{L}} \in \mathbb{R}^{1\times n}_{++}$ and $\mathbf{q}_{\mathcal{R}} \in \mathbb{R}^{n}_{++}$, respectively. Equivalently, viability ($\mathfrak{A}_3$) ensures that the system is *productive*: there exists $\mathbf{x} > \mathbf{0}$ such that $(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{y} \geqq \mathbf{0}$. A non-negative fraction of the gross product is used as means of production ($\mathbf{Ax} \in \mathbb{R}^n_+$) and the remainder constitutes net output ($\mathbf{y} \in \mathbb{R}^n_+$, with $\mathbf{y} \neq \mathbf{0}$ since $\lambda_{\mathbf{A},1} < 1$).

Wages are not advanced, so that $\mathbf{p}^T \in \mathbb{R}^{1\times n}_{++}$ satisfies (3) at uniform rate of profits $r \geq 0$ and uniform wage rate w. Equation (4) implies prices are normalised such that the value of the net output Y is the unit of account (i.e., $\mathbf{y}$ serves as *numéraire*). Equation (5) normalises the total quantity of labour employed in the economy to unity ($L = 1$); together with (4) this implies $\omega \equiv wL/Y = w$, so that the wage rate coincides with the wage share. Finally, equation (6) outlines the assumption that the profit rate r is given outside the system.

The price system (3) can be represented in terms of *vertically integrated industries* (Pasinetti, 1973, pp. 3–5). By $\mathfrak{A}_1$ and $\mathfrak{A}_3$, $(\mathbf{I} - \mathbf{A})$ is a nonsingular $\mathcal{M}$-matrix, and hence invertible with $(\mathbf{I} - \mathbf{A})^{-1} \geqq \mathbf{O}$. Given the irreducibility of $\mathbf{A}$ (see $\mathfrak{A}_2$), this inequality is strict: $(\mathbf{I} - \mathbf{A})^{-1} > \mathbf{O}$. Therefore (3) can be written as:

$$\mathbf{p}^T = r\mathbf{p}^T\mathbf{H} + w\mathbf{v}^T = w\mathbf{v}^T(\mathbf{I} - r\mathbf{H})^{-1} \tag{7}$$

where $\mathbf{v}^T = \mathbf{l}^T(\mathbf{I} - \mathbf{A})^{-1} \in \mathbb{R}^{1\times n}_{++}$ and $\mathbf{H} = \mathbf{A}(\mathbf{I} - \mathbf{A})^{-1} \in \mathbb{R}^{n\times n}_{++}$ are the vertically integrated labour-coefficients vector and input-coefficients matrix: they express the total quantities of labour and of commodity i required, respectively, to produce one unit of net output of commodity j. By (1) this also implies $\mathbf{s} = \mathbf{Ax} = \mathbf{Hy}$. From the labour normalisation (5) it follows that $L = \mathbf{l}^T\mathbf{x} = \mathbf{l}^T(\mathbf{I} - \mathbf{A})^{-1}\mathbf{y} = \mathbf{v}^T\mathbf{y} = 1$, so that the vertically integrated labour value of the social net output is also unity. Since $\mathbf{H}$ and $\mathbf{A}$ share the same eigenvectors and their eigenvalues are related by $\lambda_{\mathbf{H},i} = \frac{\lambda_{\mathbf{A},i}}{1-\lambda_{\mathbf{A},i}}$ for all $i \in N$, matrix $\mathbf{H}$ inherits $\mathfrak{A}_2$, and its P-F eigenvalue satisfies $\lambda_{\mathbf{H},1} = \frac{\lambda_{\mathbf{A},1}}{1-\lambda_{\mathbf{A},1}} > 0$.

2.2 Prices, wages, and the value of capital as functions of the rate of profits

The price system has $n + 2$ unknowns $(\mathbf{p}, w, r)$ closed by the n scalar equations in (7), the normalisation (4), and the closure (6); the labour normalisation (5) fixes the scale of the technique

$\mathcal{T}$. Let $R \equiv \frac{1}{\lambda_{\mathbf{H},1}} > 0$.[7] Then, for $r \in [0, R)$ we have that $(\mathbf{I} - r\mathbf{H})$ is a nonsingular $\mathcal{M}$-matrix that admits the Neumann expansion:

$$(\mathbf{I} - r\mathbf{H})^{-1} = \mathbf{I} + r\mathbf{H} + r^2\mathbf{H}^2 + \cdots \tag{8}$$

From (8) it is immediate that $(\mathbf{I} - r\mathbf{H})^{-1} > \mathbf{O}$ for $r \in [0, R)$. At $r = R$ the matrix $(\mathbf{I} - r\mathbf{H})$ is singular; for $r > R$ the matrix $(\frac{1}{r}\mathbf{I} - \mathbf{H})$ is nonsingular but generally has entries of mixed sign. Therefore R is the maximum rate of profit, attained at $w = 0$. For $r \in [0, R]$ we have $\mathbf{p}^T > \mathbf{0}^T$:

$$\mathbf{p}^T = \mathbf{p}^T(r) \begin{cases} = w\,\mathbf{v}^T(\mathbf{I} - r\mathbf{H})^{-1} > \mathbf{0}^T & \text{for } r \in [0, R), \\ \propto \mathbf{q}_{\mathcal{L}} > \mathbf{0}^T \text{ and } \mathbf{q}_{\mathcal{L}}\mathbf{y} = 1 & \text{for } r = R. \end{cases} \tag{9}$$

Substituting (9) into (4) and solving for w yields non-negative wages:

$$w = w(r) = \begin{cases} \dfrac{1}{\mathbf{v}^T(\mathbf{I} - r\mathbf{H})^{-1}\mathbf{y}} > 0, & r \in [0, R), \\ 0, & r = R. \end{cases} \tag{10}$$

The last equation implies that, for $r \in [0, R)$, $\mathbf{p}^T$ can be expressed as a function of r alone by substituting (10) into (9):

$$\mathbf{p}^T = \frac{1}{\mathbf{v}^T(\mathbf{I} - r\mathbf{H})^{-1}\mathbf{y}}\mathbf{v}^T(\mathbf{I} - r\mathbf{H})^{-1}. \tag{11}$$

Given (11) we can now construct aggregates of heterogeneous commodities. We define the value of the capital stock in the j-th industry per unit of net output, $\mathbf{K} \in \mathbb{R}^n$, and the economy-wide value of capital stock K:

$$\mathbf{K}^T = \mathbf{K}^T(r) = \mathbf{p}^T\mathbf{H} \tag{12}$$

$$K = K(r) = \mathbf{p}^T\mathbf{s} = \mathbf{p}^T\mathbf{A}\mathbf{x} = \mathbf{p}^T\mathbf{H}\mathbf{y}. \tag{13}$$

Using (12) and (13) we can write the industries' capital intensities and the economy-wide capital intensity as:

$$\mathbf{k}^T = \mathbf{k}^T(r) \equiv \mathbf{K}^T\hat{\mathbf{v}}^{-1}, \tag{14}$$

$$k = k(r) \equiv \frac{K}{L} = \frac{\mathbf{p}^T\mathbf{A}\mathbf{x}}{\mathbf{l}^T\mathbf{x}} = \frac{\mathbf{p}^T\mathbf{H}\mathbf{y}}{\mathbf{v}^T\mathbf{y}} = \mathbf{p}^T\mathbf{H}\mathbf{y}. \tag{15}$$

Where the last equality uses the labour normalisation $L = \mathbf{v}^T\mathbf{y} = 1$ from (5). Equation (14) gives the vertically integrated capital intensities: the proportion between the total (direct and indirect) value of means of production and the labour used in the economy to produce one unit of net output of the j-th commodity; equation (15) is the economy-wide capital intensity.

[7] The map $\lambda \mapsto \lambda/(1-\lambda)$ is continuous and strictly increasing on $[0, 1)$, with $\lim_{\lambda\to 0^+} \lambda/(1-\lambda) = 0$ and $\lim_{\lambda\to 1^-} \lambda/(1-\lambda) = +\infty$. Consequently $R = 1/\lambda_{\mathbf{H},1} \in (0, +\infty)$ for $\lambda_{\mathbf{A},1} \in (0, 1)$, with $R \to +\infty$ as $\lambda_{\mathbf{A},1} \to 0^+$ and $R \to 0^+$ as $\lambda_{\mathbf{A},1} \to 1^-$.

2.3 Properties of the curves and the standard proportions

The Neumann expansion (8) implies that for $r \in [0, R)$ the curves defined in equations (9), (10) and (13) are power series in r (and hence in general nonlinear functions of r). As $r \to R$ the curves may exhibit several inflections:

$$\mathbf{p}^T = w \sum_{m=0}^{\infty} r^m \, \mathbf{b}_m^T, \tag{16}$$

$$w = \frac{1}{\sum_{m=0}^{\infty} r^m \, b_m}, \tag{17}$$

$$K = w \sum_{m=0}^{\infty} r^m \, b_{m+1}, \tag{18}$$

where $\mathbf{b}_m^T \equiv \mathbf{v}^T \mathbf{H}^m \in \mathbb{R}_{++}^{1 \times n}$ and $b_m \equiv \mathbf{b}_m^T \mathbf{y} = \mathbf{v}^T \mathbf{H}^m \mathbf{y} \in \mathbb{R}_{++}$ $\forall m \geq 0$.[8] The wage curve (17) is monotonically decreasing in r, but its second derivative may change sign. Prices (16) and the value of the stock of means of production (18) can be either increasing or decreasing in r. At the same time, curves (16)–(18) may also bear a simple relationship with r, e.g. linear or monotonic: this depends entirely on the coefficients $\mathbf{b}_m^T$ and b_m. Because the curves are measured relative to the value of the net output (w/Y, p_j/Y, K/Y), the sources of nonlinearity can lie in either the numerator or the denominator (or both).

This means not only that K generally varies with the income distribution, but that the industry capital intensities k_j and the aggregate k need not be monotonic in r. The constraints on $\mathcal{T}$ imposed by $\mathfrak{A}_1$-$\mathfrak{A}_3$ are consistent with these two extreme cases (nonlinearities in both numerator and denominator). However, further constraints on $\mathcal{T}$ can limit the polynomial degree of the curves (16)–(18) via $\mathbf{b}_m^T$ and b_m. Two well-known constraints yield the simplest curves, i.e., a *linear* wage curve and a *constant* value of capital. We refer to the proportions of the P-F eigenvectors $\mathbf{q}_{\mathcal{R}}$ and $\mathbf{q}_{\mathcal{L}}$ as the *standard commodity* and the *standard labour* proportions, respectively.[9]

Suppose first that the social output is such that for each commodity $i \in N$ the stock of means of production $s_i = \mathbf{A}_{(i)}\mathbf{x}$ bears the same proportion to net output y_i: $s_i/y_i = \phi > 0$ $\forall i \in N$ (and hence $\mathbf{s} > \mathbf{0}$). Then $\mathbf{s} = \phi\mathbf{y}$, which by (1) and (2) implies $\mathbf{x} - \mathbf{y} = \phi\mathbf{y}$, i.e. $\mathbf{x} = (1+\phi)\mathbf{y}$ (so $\mathbf{x} \propto \mathbf{y}$). Substituting into (2) gives $\phi\mathbf{y} = \mathbf{A}(1+\phi)\mathbf{y}$, equivalently $\mathbf{A}\mathbf{y} = \frac{\phi}{1+\phi}\mathbf{y}$: $\mathbf{y}$ is a positive right eigenvector of $\mathbf{A}$. By the Perron–Frobenius theorem (see Appendix A) and $\mathfrak{A}_1$-$\mathfrak{A}_2$, $\mathbf{y}$ must correspond to the dominant eigenvalue: $\mathbf{y} \propto \mathbf{q}_{\mathcal{R}}$, with $\phi = \frac{\lambda_{\mathbf{A},1}}{1-\lambda_{\mathbf{A},1}} = \lambda_{\mathbf{H},1}$. In this case the

[8] For (18), substitute $\mathbf{p}^T = w\,\mathbf{v}^T(\mathbf{I}-r\mathbf{H})^{-1}$ into $K = \mathbf{p}^T\mathbf{H}\mathbf{y}$ and apply (8): $K = w\,\mathbf{v}^T\mathbf{H}\mathbf{y} + wr\,\mathbf{v}^T\mathbf{H}^2\mathbf{y} + \cdots = w\sum_{m=0}^{\infty} r^m\,\mathbf{v}^T\mathbf{H}^{m+1}\mathbf{y} = w\sum_{m=0}^{\infty} r^m\,b_{m+1}$. Note in particular $b_0 = \mathbf{v}^T\mathbf{y} = 1$ under the labour normalisation (5).

[9] If $\mathbf{y} \propto \mathbf{q}_{\mathcal{R}}$, then $\mathbf{q}_{\mathcal{R}}$, suitably rescaled, forms Sraffa's *standard commodity*; see Sraffa (1960) and Pasinetti (1977, Ch. 5).

Leontief inverse $(\mathbf{I}-\mathbf{A})^{-1}$ acts on $\mathbf{y}$ as a scalar: $\mathbf{x}=(\mathbf{I}-\mathbf{A})^{-1}\mathbf{y}=\frac{1}{1-\lambda_{\mathbf{A},1}}\mathbf{y}$. Equation (2) then reads $\mathbf{s}=\mathbf{A}\mathbf{x}=\phi\mathbf{y}=\lambda_{\mathbf{H},1}\mathbf{y}$ (equivalently $\mathbf{H}\mathbf{y}=\lambda_{\mathbf{H},1}\mathbf{y}$). Formally:

Assumption 2 (Standard commodity)**.** $(\mathfrak{A}_4)$ $\mathbf{y}\propto\mathbf{q}_{\mathcal{R}}$.

Under $\mathfrak{A}_4$, $\mathbf{H}\mathbf{y}=\lambda_{\mathbf{H},1}\mathbf{y}$, so $b_m=\mathbf{v}^T\mathbf{H}^m\mathbf{y}=\lambda_{\mathbf{H},1}^m\,\mathbf{v}^T\mathbf{y}=\lambda_{\mathbf{H},1}^m$ $\forall m\geq 0$, where the last equality uses the labour normalisation $\mathbf{v}^T\mathbf{y}=L=1$ established in (5). Substituting $b_m=\lambda_{\mathbf{H},1}^m$ into the geometric series (17) and (18), and using $r\lambda_{\mathbf{H},1}=r/R$, yields:

$$w\Big|_{\mathfrak{A}_4}=1-\frac{r}{R}, \tag{19}$$

$$K\Big|_{\mathfrak{A}_4}=w\sum_{m=0}^{\infty}r^m\lambda_{\mathbf{H},1}^{m+1}=w\,\frac{\lambda_{\mathbf{H},1}}{1-r/R}=\frac{1}{R}. \tag{20}$$

The value of the net output $Y=\mathbf{p}^T\mathbf{y}$ is therefore invariant to changes in the income distribution (Sraffa, 1960, Ch. IV). However, because the row vectors $\mathbf{b}_m^T=\mathbf{v}^T\mathbf{H}^m$ are not constrained by $\mathfrak{A}_4$, individual prices generally remain functions of r. Substituting $w=1-r/R$ and $\mathbf{b}_m^T=\mathbf{v}^T\mathbf{H}^m$ into (16) and collecting terms yields

$$\mathbf{p}^T\Big|_{\mathfrak{A}_4}=\mathbf{v}^T+r\,\mathbf{v}^T\left(\mathbf{H}-\lambda_{\mathbf{H},1}\mathbf{I}\right)\left(\mathbf{I}-r\mathbf{H}\right)^{-1}. \tag{21}$$

Equation (21) makes explicit that prices are constant in r if and only if $\mathbf{v}^T\left(\mathbf{H}-\lambda_{\mathbf{H},1}\mathbf{I}\right)=\mathbf{0}^T$, that is, if and only if $\mathbf{v}^T$ (equivalently $\mathbf{l}^T$) is the left P-F eigenvector of $\mathbf{H}$ (resp. $\mathbf{A}$): $\mathbf{v}^T\mathbf{H}=\lambda_{\mathbf{H},1}\mathbf{v}^T$. This is the dual *standard labour* condition $\mathbf{v}\propto\mathbf{q}_{\mathcal{L}}$.

A sufficient economic condition for $\mathbf{v}\propto\mathbf{q}_{\mathcal{L}}$ is the equality of vertically integrated capital intensities across industries. Suppose now that for each industry $j\in N$ the vertically integrated capital intensity $k_j=K_j/v_j$ takes the same value across industries: $k_j=\kappa>0$ $\forall j\in N$, that is to say $\mathbf{k}^T=\kappa\,\mathbf{e}^T$, which by (14) and the definition of $\hat{\mathbf{v}}$ is equivalent to $\mathbf{p}^T\mathbf{H}=\kappa\,\mathbf{v}^T$ (i.e., $\mathbf{K}^T\propto\mathbf{v}^T$). Substituting into the vertically integrated price system (7) gives $\mathbf{p}^T=(r\kappa+w)\,\mathbf{v}^T$, so prices are proportional to vertically integrated labour values: $\mathbf{p}^T\propto\mathbf{v}^T$. Post-multiplying the latter by $\mathbf{H}$ and using $\mathbf{p}^T\mathbf{H}=\kappa\,\mathbf{v}^T$ yields $\mathbf{v}^T\mathbf{H}=\frac{\kappa}{r\kappa+w}\,\mathbf{v}^T$: $\mathbf{v}^T$ is a positive left eigenvector of $\mathbf{H}$. By the Perron–Frobenius theorem (see Appendix A) and under $\mathfrak{A}_1$-$\mathfrak{A}_2$, $\mathbf{v}^T$ must be the dominant one: $\mathbf{v}^T\propto\mathbf{q}_{\mathcal{L}}$, with eigenvalue $\frac{\kappa}{r\kappa+w}=\lambda_{\mathbf{H},1}$, so that $\kappa=\frac{w\,\lambda_{\mathbf{H},1}}{1-r\,\lambda_{\mathbf{H},1}}=\frac{w}{R-r}$.

Note that, since $\mathbf{v}^T=\mathbf{l}^T(\mathbf{I}-\mathbf{A})^{-1}$ and $\mathbf{H}$ and $\mathbf{A}$ share left eigenvectors, this is equivalent to $\mathbf{l}^T\propto\mathbf{q}_{\mathcal{L}}$: the labour-coefficients vector is aligned with the left P-F eigenvector of $\mathbf{A}$. Note that (12) then reduces to $\mathbf{K}^T=\kappa\,\mathbf{v}^T$ (equivalently, $\mathbf{v}^T\mathbf{H}=\lambda_{\mathbf{H},1}\mathbf{v}^T$, the left-eigenvalue equation for $\mathbf{H}$). Conversely, if $\mathbf{v}^T\propto\mathbf{q}_{\mathcal{L}}$, then $\mathbf{v}^T\mathbf{H}=\lambda_{\mathbf{H},1}\mathbf{v}^T$ and prices satisfy $\mathbf{p}^T=(r\kappa+w)\mathbf{v}^T\propto\mathbf{v}^T$ for some scalar $r\kappa+w$, so that $\mathbf{K}^T=\mathbf{p}^T\mathbf{H}=(r\kappa+w)\lambda_{\mathbf{H},1}\mathbf{v}^T\propto\mathbf{v}^T$, giving $\mathbf{k}^T=\kappa\mathbf{e}^T$ for $\kappa=(r\kappa+w)\lambda_{\mathbf{H},1}$, i.e. $\kappa=w/(R-r)$. Equal vertically integrated capital intensities is therefore equivalent to $\mathbf{v}\propto\mathbf{q}_{\mathcal{L}}$. Formally, the assumption reads:

Assumption 3 (Standard labour)**.** ($\mathfrak{A}_5$) $\mathbf{v} \propto \mathbf{q}_{\mathcal{L}}$.

Under $\mathfrak{A}_5$, $\mathbf{v}^T\mathbf{H} = \lambda_{\mathbf{H},1}\mathbf{v}^T$, so the row vectors in the price expansion collapse to a geometric sequence: $\mathbf{b}_m^T = \mathbf{v}^T\mathbf{H}^m = \lambda_{\mathbf{H},1}^m\mathbf{v}^T$ and, post-multiplying by $\mathbf{y}$ and using $\mathbf{v}^T\mathbf{y} = 1$, $b_m = \lambda_{\mathbf{H},1}^m$ $\forall m \geq 0$. Substituting $\mathbf{b}_m^T = \lambda_{\mathbf{H},1}^m\mathbf{v}^T$ and $b_m = \lambda_{\mathbf{H},1}^m$ into (16)–(18) and summing the geometric series yields

$$w\Big|_{\mathfrak{A}_5} = \frac{1}{\sum_{m=0}^{\infty}(r\lambda_{\mathbf{H},1})^m} = 1 - \frac{r}{R}, \tag{22}$$

$$\mathbf{p}^T\Big|_{\mathfrak{A}_5} = w\,\mathbf{v}^T\sum_{m=0}^{\infty}(r\lambda_{\mathbf{H},1})^m = \frac{w}{1-r/R}\mathbf{v}^T = \mathbf{v}^T, \tag{23}$$

$$K\Big|_{\mathfrak{A}_5} = w\sum_{m=0}^{\infty}r^m\lambda_{\mathbf{H},1}^{m+1} = \frac{w\lambda_{\mathbf{H},1}}{1-r/R} = \frac{1}{R}. \tag{24}$$

Where in (23) we used $w = 1 - r/R$ from (22). Assumptions $\mathfrak{A}_4$ ($\mathbf{y} \propto \mathbf{q}_{\mathcal{R}}$, equivalent to equal stock-to-net-output proportions across commodities) and $\mathfrak{A}_5$ ($\mathbf{v} \propto \mathbf{q}_{\mathcal{L}}$, equivalent to equal vertically integrated capital intensities across industries) are particular cases of more general conditions yielding (23)–(24). Further nonzero-slope linear price and wage curves, and their associated constraints on the structure of technology and demand $\mathcal{T}$, have been identified via the spectral representation of (16)–(18), which is discussed in the next section.

2.4 Spectral representation of the curves

Consider the spectrum of matrix $\mathbf{A}$. Let $e \in (0, n]$ be the number of distinct eigenvalues of $\mathbf{A}$, ordered such that $\lambda_{\mathbf{A},1} > |\lambda_{\mathbf{A},k}|$ $\forall k = 2, 3, ..., e$, where $\lambda_{\mathbf{A},k} \in \mathbb{C}$ for $k = 2, 3, ..., e$ are the subdominant eigenvalues of $\mathbf{A}$. Let $\epsilon_k \in [1, n]$ be the *algebraic multiplicity* of $\lambda_{\mathbf{A},k}$ (i.e., $\lambda_{\mathbf{A},k}$ repeats itself ϵ_k times as an eigenvalue of $\mathbf{A}$: $\sum_{k=1}^{e}\epsilon_k = n$). Let $\varepsilon_k \in [1, \epsilon_k]$ be the *geometric multiplicity* of $\lambda_{\mathbf{A},k}$, i.e., the number of linearly independent eigenvectors associated with $\lambda_{\mathbf{A},k}$, which form a rectangular matrix $\mathbf{W}_k \in \mathbb{C}^{n\times\varepsilon_k}$ that has as columns the linearly independent right-hand eigenvectors associated with $\lambda_{\mathbf{A},k}$. Analogously, we have that $\mathbf{Z}_k \in \mathbb{C}^{n\times\varepsilon_k}$ is the rectangular matrix that has as rows the linearly independent left-hand eigenvectors associated with $\lambda_{\mathbf{A},k}$.

This means of course that $\mathbf{A}\mathbf{W}_k = \lambda_{\mathbf{A},k}\mathbf{W}_k$ and $\mathbf{Z}_k\mathbf{A} = \mathbf{Z}_k\lambda_{\mathbf{A},k}$ for $k = 1, 2, ..., e$. Let $\mathbf{Q} \equiv [\mathbf{W}_1|\mathbf{W}_2|\cdots|\mathbf{W}_e]$ and $\mathbf{Q}^{-1} \equiv [\mathbf{Z}_1|\mathbf{Z}_2|\cdots|\mathbf{Z}_e]^T$ i.e., $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}_{\mathbf{A}}\mathbf{Q}^{-1}$, where $\mathbf{\Lambda}_{\mathbf{A}} \equiv \text{diag}\{\lambda_{\mathbf{A},1}, \lambda_{\mathbf{A},2}, ..., \lambda_{\mathbf{A},n}\}$. Note that by $\mathfrak{A}_2$, we know that $\mathbf{A}$ has a set of n linearly independent eigenvectors: $\epsilon_k = \varepsilon_k$ $\forall k = 1, 2, ..., e$: $\mathbf{Q} \in \mathbb{C}^{n\times n}$ has a full rank (in general $\mathbf{Q}$ is complex-valued, since the subdominant eigenvalues of a primitive non-negative matrix may be complex). Where $\mathbf{Q}$ is a matrix that has as columns the right-hand eigenvectors $\mathbf{q}_{\mathcal{R},t}$ and $\mathbf{Q}^{-1}$ has as rows the left-hand eigenvectors $\mathbf{q}_{\mathcal{L},t}^T$ $\forall t = 1, 2, ..., n$: hence the right-hand and left-hand P-F eigenvectors are $\mathbf{q}_{\mathcal{R},1} = \mathbf{q}_{\mathcal{R}}$ and

$\mathbf{q}^T_{\mathcal{L},1} = \mathbf{q}_{\mathcal{L}}$ respectively.[10] Vectors $\mathbf{q}_{\mathcal{R},t}$ and $\mathbf{q}^T_{\mathcal{L},t}$, for $t = 1, 2, ..., n$ constitute a basis, any column $\mathbf{c}$ and row $\mathbf{r}$ vector can be expressed as $\mathbf{c} = \mathbf{Q}\mathbf{c}^*$ and $\mathbf{r} = \mathbf{r}^*\mathbf{Q}^{-1}$. Hence, with $\mathbf{Q}$ and $\mathbf{Q}^{-1}$ we can express the vertically integrated technique of production $\mathcal{V} = \{(\mathbf{H}, \mathbf{v}), \mathbf{y}\}$ in the coordinate space given by the eigenvectors of matrix $\mathbf{A}$:

$$\mathbf{H} = \mathbf{Q}\boldsymbol{\Lambda}_{\mathbf{H}}\mathbf{Q}^{-1} \qquad \Rightarrow \boldsymbol{\Lambda}_{\mathbf{H}} = \mathbf{Q}^{-1}\mathbf{H}\mathbf{Q}, \tag{25}$$

$$\mathbf{v}^T = \sum_{k=1}^{e} \boldsymbol{\alpha}_k^T \mathbf{Z}_k^T = \boldsymbol{\alpha}^T\mathbf{Q}^{-1} \qquad \Rightarrow \boldsymbol{\alpha}^T = \mathbf{v}^T\mathbf{Q} = [\boldsymbol{\alpha}_k^T], \tag{26}$$

$$\mathbf{y} = \sum_{k=1}^{e} \mathbf{W}_k \boldsymbol{\chi}_k = \mathbf{Q}\boldsymbol{\chi} \qquad \Rightarrow \boldsymbol{\chi} = \mathbf{Q}^{-1}\mathbf{y} = [\chi_k]. \tag{27}$$

Where $\boldsymbol{\Lambda}_{\mathbf{A}}(\mathbf{I} - \boldsymbol{\Lambda}_{\mathbf{A}})^{-1} = \boldsymbol{\Lambda}_{\mathbf{H}} \equiv \text{diag}\{\lambda_{\mathbf{H},1}, \lambda_{\mathbf{H},2}, ..., \lambda_{\mathbf{H},n}\}$, $\boldsymbol{\alpha}_k^T = \mathbf{v}^T\mathbf{W}_k \in \mathbb{C}^{1\times\varepsilon_k}$ and $\boldsymbol{\chi}_k = \mathbf{Z}_k^T\mathbf{y} \in \mathbb{C}^{\epsilon_k}$ for $k = 2, 3, ..., e$. For $k = 1$, due to the irreducibility assumption outlined in $\mathfrak{A}_2$, we know that $\lambda_{\mathbf{H},1}$ is a simple eigenvalue with $\epsilon_1 = \varepsilon_1 = 1$, therefore $\mathbf{W}_1 = \mathbf{q}_{\mathcal{R},1} = \mathbf{q}_{\mathcal{R}}$ and $\mathbf{Z}_1^T = \mathbf{q}^T_{\mathcal{L},1} = \mathbf{q}_{\mathcal{L}}$. Furthermore $\boldsymbol{\alpha}_1^T = \mathbf{v}^T\mathbf{W}_1 = \mathbf{v}^T\mathbf{q}_{\mathcal{R}} = \alpha_1 \in \mathbb{R}_+$ and $\boldsymbol{\chi}_1 = \mathbf{Z}_1^T\mathbf{y} = \mathbf{q}_{\mathcal{L}}\mathbf{y} = \chi_1 = 1$.[11] Stacking the blocks $\{\boldsymbol{\alpha}_k^T\}_{k=1}^e$ horizontally yields the full row vector $\boldsymbol{\alpha}^T = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^{1\times n}$, and stacking the blocks $\{\boldsymbol{\chi}_k\}_{k=1}^e$ vertically yields the full column vector $\boldsymbol{\chi} = (\chi_1 = 1, \chi_2, \ldots, \chi_n)^T \in \mathbb{C}^n$ (under $\mathfrak{A}_2$ both stackings have dimension $\sum_{k=1}^e \epsilon_k = n$).

We refer to $\boldsymbol{\alpha}^T$ and $\boldsymbol{\chi}$ as the eigenlabour-vector and the eigenoutput-vector, and to their coordinates $\alpha_1, \alpha_2, ..., \alpha_n$ and $\chi_1, \chi_2, ..., \chi_n$ as the *eigenlabours* and *eigenoutputs*, respectively. The block products $\boldsymbol{\alpha}_k^T\boldsymbol{\chi}_k \in \mathbb{C}$ (the inner products between the projection of $\mathbf{v}^T$ onto the left eigenspace of $\lambda_{\mathbf{A},k}$ and the projection of $\mathbf{y}$ onto the right eigenspace of $\lambda_{\mathbf{A},k}$) are scalars for each k, regardless of the multiplicity ϵ_k. For the purpose of analyzing wage-profit curves, we substitute (25)–(27) into (17):

$$b_m = \mathbf{v}^T\mathbf{H}^m\mathbf{y} = \mathbf{v}^T\mathbf{Q}\boldsymbol{\Lambda}_{\mathbf{H}}^m\mathbf{Q}^{-1}\mathbf{y} = \boldsymbol{\alpha}^T\boldsymbol{\Lambda}_{\mathbf{H}}^m\boldsymbol{\chi} = \lambda_{\mathbf{H},1}^m\alpha_1\chi_1 + \sum_{k=2}^{e}\lambda_{\mathbf{H},k}^m\boldsymbol{\alpha}_k^T\boldsymbol{\chi}_k. \tag{28}$$

Substituting (28) into (17) we can obtain the following equation:

$$w(r) = \frac{1}{\sum_{m=0}^{\infty} r^m(\lambda_{\mathbf{H},1}^m\alpha_1\chi_1 + \sum_{k=2}^{e}\lambda_{\mathbf{H},k}^m\boldsymbol{\alpha}_k^T\boldsymbol{\chi}_k)}. \tag{29}$$

Alternatively, we can substitute (25) into (10) to get:

$$w(r) = \frac{1}{\boldsymbol{\alpha}^T[\mathbf{I} - r\boldsymbol{\Lambda}_{\mathbf{H}}]\boldsymbol{\chi}} = \frac{1}{\sum_{k=1}^{e}\frac{\boldsymbol{\alpha}_k^T\boldsymbol{\chi}_k}{1-r\lambda_{\mathbf{H},k}}} = \frac{1}{\frac{\alpha_1\chi_1}{1-r\lambda_{\mathbf{H},1}} + \sum_{k=2}^{e}\frac{\boldsymbol{\alpha}_k^T\boldsymbol{\chi}_k}{1-r\lambda_{\mathbf{H},k}}}. \tag{30}$$

[10] We fix the scale of the eigenvectors as follows. The left-hand P-F eigenvector $\mathbf{q}_{\mathcal{L}}$ inherits the normalisation $\mathbf{q}_{\mathcal{L}}\mathbf{y} = 1$ from (10) (i.e., $\mathbf{q}_{\mathcal{L}}$ is the limiting price vector at $r = R$, expressed in units of the numéraire $\mathbf{y}$). The right-hand P-F eigenvector $\mathbf{q}_{\mathcal{R}}$ is then scaled so that $\mathbf{q}_{\mathcal{L}}\mathbf{q}_{\mathcal{R}} = 1$, and the remaining left- and right-hand eigenvectors $(\mathbf{q}^T_{\mathcal{L},t}, \mathbf{q}_{\mathcal{R},t})$ for $t = 2, \ldots, n$ are paired biorthogonally so that $\mathbf{q}^T_{\mathcal{L},t}\mathbf{q}_{\mathcal{R},s} = \delta_{ts}$ for all $t, s \in N$, ensuring $\mathbf{Q}^{-1}\mathbf{Q} = \mathbf{I}$.

[11] Recall the normalisation convention adopted in the footnote above: $\mathbf{q}_{\mathcal{L}}\mathbf{y} = 1$.

Equations (29)–(30) show that $w(r)$ also depends on: (i) the structural relationship that $\mathbf{H}$ has with vectors $\mathbf{v}^T$ and $\mathbf{y}$; and (ii) the inner structure of $\mathbf{H}$, independent of the coordinate space. For $t = 1, 2, ..., n$, we can see that the former is captured by the eigenlabours α_t and eigenoutputs χ_t (contained in their respective vectors $\boldsymbol{\alpha}^T$ and $\boldsymbol{\chi}$), whereas the latter is represented by its eigenvalues $\lambda_{\mathbf{H},t}$. The non-linear behaviour of (29)–(30) will the depend on the accumulation of the e terms by $\sum_{k=2}^{e} \lambda_{\mathbf{H},k}^{m} \boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k$ and $\sum_{k=1}^{e} \frac{\boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k}{1 - r\lambda_{\mathbf{H},k}}$, i.e., the closer to zero the eigenlabours α_t, eigenoutputs χ_t and eigenvalues $\lambda_{\mathbf{H},t}$ are, the lesser the degree of non-linearity produced by the term.

2.5 General constraints for the simple curves

The spectral representations described in (29)–(30) isolate the structural origins of the potential non-linearities of wage profit curves $w(r)$. As seen in (29), we contend that each subdominant eigenvalue $\lambda_{\mathbf{H},k}$ for $k \geq 2$ contributes a term whose weight is the scalar inner product $\boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k$. We collect this observation formally.

Proposition 1 (Spectral characterisation of linearity)**.** *Under $\mathfrak{A}_1$–$\mathfrak{A}_3$, the wage curve $w(r)$ is exactly linear in r on $[0, R]$, with $w(r) = 1 - r/R$, if and only if*

$$\boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k = 0 \quad \textit{for all } k = 2, 3, \ldots, e. \tag{31}$$

Equivalently, $b_m = \lambda_{\mathbf{H},1}^{m}$ for all $m \geq 0$, and the value of the capital stock $K(r)$ is constant at $K = 1/R$.

Proof. ($\Leftarrow$) Assume (31). Then (28) reduces to $b_m = \lambda_{\mathbf{H},1}^{m}\, \alpha_1 \chi_1$ for all $m \geq 0$. Setting $m = 0$, $b_0 = \alpha_1 \chi_1 = \mathbf{v}^T \mathbf{y} = 1$ by (5), hence $\alpha_1 \chi_1 = 1$ and $b_m = \lambda_{\mathbf{H},1}^{m}$ for all $m \geq 0$. Substituting into (17) and summing the geometric series gives $w(r) = 1 - r/R$ on $[0, R]$. Substituting into (18) gives $K(r) = w\, \lambda_{\mathbf{H},1}/(1 - r/R) = 1/R$. ($\Rightarrow$) If $w(r)$ is linear on $[0, R]$ with $w(0) = 1$ and $w(R) = 0$, then $w(r) = 1 - r/R$, hence $\sum_{m \geq 0} r^m b_m = 1/(1 - r\lambda_{\mathbf{H},1}) = \sum_{m \geq 0} (r\lambda_{\mathbf{H},1})^m$ on $[0, R)$. Matching coefficients gives $b_m = \lambda_{\mathbf{H},1}^{m}$ for all $m \geq 0$. Subtracting from (28) and using $\alpha_1 \chi_1 = b_0 = 1$ yields $\sum_{k=2}^{e} \lambda_{\mathbf{H},k}^{m} \boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k = 0$ for all $m \geq 0$. Since the subdominant eigenvalues $\{\lambda_{\mathbf{H},k}\}_{k=2}^{e}$ are distinct, the corresponding Vandermonde system has full rank, hence $\boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k = 0$ for all $k \geq 2$. □

Proposition 1 reduces the problem of exact linearity to a set of $e - 1$ scalar orthogonality conditions in the eigenbasis of $\mathbf{A}$. Two structural mechanisms make each condition hold for all $k \geq 2$ simultaneously: either the projection of $\mathbf{v}^T$ on the left eigenspace of $\lambda_{\mathbf{A},k}$ vanishes ($\boldsymbol{\alpha}_k = \mathbf{0}$), or the projection of $\mathbf{y}$ on the right eigenspace of $\lambda_{\mathbf{A},k}$ vanishes ($\boldsymbol{\chi}_k = \mathbf{0}$). These two sufficient conditions recover, respectively, the standard-labour and standard-commodity assumptions.

Corollary 1 ($\mathfrak{A}_5$ as a spectral condition)**.** *Under $\mathfrak{A}_1$–$\mathfrak{A}_3$ the following are equivalent:*

(i) $\mathfrak{A}_5$ holds: $\mathbf{v} \propto \mathbf{q}_{\mathcal{L}}$.

(ii) $\boldsymbol{\alpha}_k = \mathbf{0}$ for all $k = 2, \ldots, e$, i.e. all subdominant eigenlabours vanish.

Under either condition, $\boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k = 0$ for all $k \geq 2$, and Proposition 1 applies. Moreover, prices are constant in r: $\mathbf{p}^T(r) = \mathbf{v}^T$.

Proof. (i)⇒(ii): If $\mathbf{v}^T = \sigma \mathbf{q}_{\mathcal{L}}$ for some $\sigma > 0$, then by the biorthogonality of $\mathbf{Q}^{-1}\mathbf{Q} = \mathbf{I}$, $\boldsymbol{\alpha}_k^T = \mathbf{v}^T \mathbf{W}_k = \sigma \mathbf{q}_{\mathcal{L}} \mathbf{W}_k = \sigma \mathbf{q}_{\mathcal{L},1}^T \mathbf{W}_k = \mathbf{0}^T$ for all $k \geq 2$, since $\mathbf{W}_k$ is composed of right eigenvectors orthogonal to $\mathbf{q}_{\mathcal{L},1}$. (ii)⇒(i): If $\boldsymbol{\alpha}_k = \mathbf{0}$ for $k \geq 2$, then $\mathbf{v}^T = \boldsymbol{\alpha}^T \mathbf{Q}^{-1} = \alpha_1 \mathbf{q}_{\mathcal{L},1}^T = \alpha_1 \mathbf{q}_{\mathcal{L}}$ by (26). The price statement follows from (23). □

Corollary 2 ($\mathfrak{A}_4$ as a spectral condition)**.** *Under $\mathfrak{A}_1$–$\mathfrak{A}_3$ the following are equivalent:*

(i) $\mathfrak{A}_4$ holds: $\mathbf{y} \propto \mathbf{q}_{\mathcal{R}}$.

(ii) $\boldsymbol{\chi}_k = \mathbf{0}$ for all $k = 2, \ldots, e$, i.e. all subdominant eigenoutputs vanish.

Under either condition, $\boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k = 0$ for all $k \geq 2$, and Proposition 1 applies. Moreover, the gross output is also aligned: $\mathbf{x} \propto \mathbf{q}_{\mathcal{R}}$.

Proof. (i)⇔(ii): symmetric to Corollary 1, using $\mathbf{y} = \mathbf{Q}\boldsymbol{\chi} = \chi_1 \mathbf{q}_{\mathcal{R},1} + \sum_{k \geq 2} \mathbf{W}_k \boldsymbol{\chi}_k$. For the second claim, $\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{y}$ and $\mathbf{q}_{\mathcal{R}}$ is an eigenvector of $(\mathbf{I} - \mathbf{A})^{-1}$ with eigenvalue $(1 - \lambda_{\mathbf{A},1})^{-1}$, hence $\mathbf{x} \propto \mathbf{q}_{\mathcal{R}}$ whenever $\mathbf{y} \propto \mathbf{q}_{\mathcal{R}}$. □

Corollaries 1 and 2 make explicit that $\mathfrak{A}_5$ and $\mathfrak{A}_4$ are the two "corner" ways of satisfying (31): they suppress the subdominant terms in (28) by zeroing one of the two factors of each product $\boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k$. They are not the only ones. More generally, (31) can hold through cancellation between non-zero $\boldsymbol{\alpha}_k$ and non-zero $\boldsymbol{\chi}_k$, or through any combination of selective vanishing on subsets of the subdominant spectrum.

We emphasise an asymmetry between the two corner cases that the parallel structure of Corollaries 1 and 2 can obscure: under $\mathfrak{A}_5$ the price vector $\mathbf{p}^T(r)$ is constant in r and equal to $\mathbf{v}^T$, whereas under $\mathfrak{A}_4$ the wage curve is linear and K is constant but prices generally remain functions of r (21). We argue that $\mathfrak{A}_5$ is the strictly stronger structural condition. This distinction matters for the study of empirical regularities: the alignment of $\mathbf{l}$ with $\mathbf{q}_{\mathcal{L}}$ implies near-constancy of relative prices, while the alignment of $\mathbf{y}$ (and $\mathbf{x}$) with $\mathbf{q}_{\mathcal{R}}$ does not.

Equivalences are summarised in Table 1, which restates Corollaries 1 and 2 in matched algebraic form. The two columns are parallel up to the price-constancy asymmetry noted above.

Table 1: Spectral characterisation of the standard-labour and standard-commodity conditions.

	Standard labour ($\mathfrak{A}_5$) $\mathbf{l}^T \propto \mathbf{q}_{\mathcal{L}}$ (equiv. $\mathbf{v}^T \propto \mathbf{q}_{\mathcal{L}}$)	Standard commodity ($\mathfrak{A}_4$) $\mathbf{y} \propto \mathbf{q}_{\mathcal{R}}$ (equiv. $\mathbf{x} \propto \mathbf{q}_{\mathcal{R}}$)
Spectral form	$\boldsymbol{\alpha}_k^T = \mathbf{0}^T \ \forall k \geq 2$	$\boldsymbol{\chi}_k = \mathbf{0} \ \forall k \geq 2$
Eigen-equation	$\mathbf{v}^T\mathbf{H} = \lambda_{\mathbf{H},1}\mathbf{v}^T$	$\mathbf{H}\mathbf{y} = \lambda_{\mathbf{H},1}\mathbf{y}$
Proportions of $\mathbf{H}$	$\mathbf{k}^T = \kappa\, \mathbf{e}^T$	$\mathbf{s} = \phi\, \mathbf{y}$
	$\kappa = w/(R-r)$	$\phi = \lambda_{\mathbf{H},1}$
Wage curve	$w(r) = 1 - r/R$	$w(r) = 1 - r/R$
Aggregate capital	$K(r) = 1/R$	$K(r) = 1/R$
Prices	$\mathbf{p}^T(r) = \mathbf{v}^T$, constant	$\mathbf{p}^T(r)$ generally non-constant

2.5.1 Near-linearity: three structural channels

Proposition 1 characterises exact linearity. Empirically, exact alignment of $\mathbf{v}^T$ with $\mathbf{q}_{\mathcal{L}}$ or of $\mathbf{y}$ with $\mathbf{q}_{\mathcal{R}}$ is not to be expected. The relevant notion is therefore that of *near*-linearity, quantified by the size of the subdominant inner products

$$\sigma_k \;\equiv\; \boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k \;\in\; \mathbb{C}, \qquad k = 2, 3, \ldots, e. \tag{32}$$

The labour normalisation $b_0 = \mathbf{v}^T\mathbf{y} = 1$ together with the spectral expansion (28) at $m = 0$ gives the identity $\alpha_1\chi_1 + \sum_{k=2}^{e} \sigma_k = 1$, so that $\alpha_1\chi_1 = 1 - \sum_{k=2}^{e} \sigma_k$ is determined by the subdominant inner products. Substituting this last expression back into (28) yields the decomposition

$$b_m \;=\; \lambda_{\mathbf{H},1}^m \;+\; \tilde{\rho}_m, \qquad \tilde{\rho}_m \;\equiv\; \sum_{k=2}^{e} \sigma_k \left(\lambda_{\mathbf{H},k}^m - \lambda_{\mathbf{H},1}^m\right), \tag{33}$$

which expresses the moment sequence as the geometric coefficient $\lambda_{\mathbf{H},1}^m$ corresponding to exact linearity, plus a remainder $\tilde{\rho}_m$ that vanishes for all m if and only if (31) holds. The remainder is bounded by

$$|\tilde{\rho}_m| \;\leq\; \sum_{k=2}^{e} |\sigma_k| \big(|\lambda_{\mathbf{H},k}|^m + \lambda_{\mathbf{H},1}^m\big). \tag{34}$$

For the wage curve itself, substituting $\alpha_1\chi_1 = 1 - \sum_{k=2}^{e} \sigma_k$ into (30) and simplifying gives the exact identity

$$w(r) \;=\; \frac{1 - r/R}{1 \;+\; r \displaystyle\sum_{k=2}^{e} \frac{\sigma_k \left(\lambda_{\mathbf{H},k} - \lambda_{\mathbf{H},1}\right)}{1 - r\,\lambda_{\mathbf{H},k}}}, \qquad r \in [0, R]. \tag{35}$$

The denominator equals one at $r = 0$ and stays bounded on $[0, R]$ by the distinctness $|\lambda_{\mathbf{H},k}| < \lambda_{\mathbf{H},1} = 1/R$. The wage curve coincides with $1 - r/R$ on $[0, R]$ if and only if the sum in the denominator of (35) vanishes for all $r \in [0, R)$, which by analyticity and the distinctness of the subdominant eigenvalues is equivalent to $\sigma_k = 0$ for all $k \geq 2$, recovering Proposition 1. Expressions (34) and (35) reveal three distinct structural channels through which the structure of technology and demand captured by $\mathcal{T}$ delivers a near-linear wage curve, corresponding to three independent mechanisms by which $|\sigma_k|$ or its impact may be small for all $k \geq 2$:

(C1) *Labour channel.* $\|\boldsymbol{\alpha}_k\|$ small for all $k \geq 2$, corresponding to $\mathbf{v}^T \approx \alpha_1 \, \mathbf{q}_{\mathcal{L}}$ (equivalently $\mathbf{l}^T$ approximately aligned with the left P-F eigenvector of $\mathbf{A}$). Since $|\sigma_k| \leq \|\boldsymbol{\alpha}_k\| \, \|\boldsymbol{\chi}_k\|$ by Cauchy–Schwarz, this suppresses every subdominant inner product. A partial realisation of Corollary 1.

(C2) *Output channel.* $\|\boldsymbol{\chi}_k\|$ small for all $k \geq 2$, corresponding to $\mathbf{y} \approx \chi_1 \, \mathbf{q}_{\mathcal{R}}$ (equivalently $\mathbf{x} \approx \mathbf{q}_{\mathcal{R}}$, by Corollary 2). Same Cauchy–Schwarz mechanism. A partial realisation of Corollary 2.

(C3) *Spectral channel.* Subdominant eigenvalues $|\lambda_{\mathbf{H},k}|$ small for all $k \geq 2$, i.e. a large spectral gap $\lambda_{\mathbf{H},1}/|\lambda_{\mathbf{H},2}| \gg 1$ and a subdominant spectrum clustered near zero. This channel does not bound $|\sigma_k|$ itself but suppresses its dynamical contribution: in (33), $|\lambda_{\mathbf{H},k}|^m \to 0$ geometrically for $m \geq 1$, so even non-vanishing σ_k produces only small curvature; in (35), the factor $(\lambda_{\mathbf{H},k} - \lambda_{\mathbf{H},1})/(1 - r\lambda_{\mathbf{H},k})$ is bounded in modulus by approximately $\lambda_{\mathbf{H},1}$ when $|\lambda_{\mathbf{H},k}| \ll \lambda_{\mathbf{H},1}$, capping each subdominant contribution.

The three channels are not mutually exclusive and can reinforce one another. Channels C1 and C2 act on the *geometry* of the technique, i.e., the alignment of the labour vector and the demand vector with the dominant eigenspaces of $\mathbf{A}$. These are properties of the pair $(\mathbf{v}, \mathbf{y})$ relative to $\mathbf{A}$. Channel C3 is a property of $\mathbf{A}$ alone, independent of $\mathbf{l}$ and $\mathbf{y}$: it is the *internal* spectral structure of the input–output matrix. If any single channel is small, the deviation of $w(r)$ from $1 - r/R$ is small; if more than one is small simultaneously, the deviation is small to second order. The three empirical regularities (i)–(iii) reported in the abstract are quantitative versions of these three channels: regularity (i) bounds $\|\boldsymbol{\alpha}_k\|$ (C1), regularity (ii) bounds $\|\boldsymbol{\chi}_k\|$ (C2), and regularity (iii) bounds $|\lambda_{\mathbf{H},k}|$ (C3).

3 Empirical Evidence from U.S. Input–Output Accounts

This section evaluates the three structural channels (C1)–(C3) of Section 2 using the U.S. Benchmark Input–Output Accounts (BEA, 2022, 2025), covering the years 1977, 1982, 1987, 1992, 1997, 2002, 2007, 2012 and 2017, with sectoral disaggregation ranging from $n = 399$ to $n = 528$. For each benchmark year we recover the technique $\mathcal{T} = \{(\mathbf{A}, \mathbf{l}); \mathbf{x}\}$, compute the vertically integrated representation $\mathcal{V} = \{(\mathbf{H}, \mathbf{v}); \mathbf{y}\}$, and the spectral decomposition $\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}_{\mathbf{A}}\mathbf{Q}^{-1}$, together with the eigenlabours $\boldsymbol{\alpha}^T = \mathbf{v}^T\mathbf{Q}$ and the eigenoutputs $\boldsymbol{\chi} = \mathbf{Q}^{-1}\mathbf{y}$ under the normalisations established in Section 2. Throughout, we treat the cross-sectoral distributions of the resulting quantities as the object of interest: the regularities reported below are statistical properties of the technique, exhibiting both robust central tendencies and non-trivial heterogeneities. We organise the evidence around the three channels of Section 2.5.1: §3.1 documents the labour and output channels (C1) and (C2); §3.2 documents the spectral channel (C3) and the joint collapse of the eigenlabour and eigenoutput coordinates onto the dominant eigenspaces.

3.1 Alignment of $\mathbf{v}^T$ and $\mathbf{y}$ with the Perron–Frobenius eigenvectors

Corollaries 1 and 2 characterised the standard-labour and standard-commodity conditions as *alignments* of the vertically integrated labour vector $\mathbf{v}^T$ with $\mathbf{q}_{\mathcal{L}}$ and of the net output $\mathbf{y}$ with $\mathbf{q}_{\mathcal{R}}$, respectively. We can derive two types of statements: (i) a *geometric* statement (proportionality with the relevant P-F eigenvector); and (ii) a *dynamic* statement (the vector is fixed by the relevant eigen-equation of $\mathbf{H}$). We measure deviations from each of the four equivalent equalities through the residual vectors:

$$\boldsymbol{\eta}^T = \mathbf{v}^T - \alpha_1\,\mathbf{q}_{\mathcal{L}}, \qquad \boldsymbol{\xi}^T = \mathbf{v}^T\mathbf{H} - \lambda_{\mathbf{H},1}\,\mathbf{v}^T, \tag{36}$$

$$\boldsymbol{\mu} = \mathbf{y} - \chi_1\,\mathbf{q}_{\mathcal{R}}, \qquad \boldsymbol{\zeta} = \mathbf{H}\mathbf{y} - \lambda_{\mathbf{H},1}\,\mathbf{y}, \tag{37}$$

each of which vanishes identically under the corresponding exact alignment ($\boldsymbol{\eta}^T = \boldsymbol{\xi}^T = \mathbf{0}^T$ under $\mathbf{v} \propto \mathbf{q}_{\mathcal{L}}$; $\boldsymbol{\mu} = \boldsymbol{\zeta} = \mathbf{0}$ under $\mathbf{y} \propto \mathbf{q}_{\mathcal{R}}$). The two formulations of each channel are not mutually redundant in finite samples: $\boldsymbol{\eta}^T$ and $\boldsymbol{\mu}$ measure proximity of the empirical vectors to a one-dimensional subspace (geometry), while $\boldsymbol{\xi}^T$ and $\boldsymbol{\zeta}$ measure how nearly the empirical vectors are fixed under the action of $\mathbf{H}$ (dynamics). Reporting both guards against artefacts of the particular normalisation adopted for $\mathbf{q}_{\mathcal{L}}$ and $\mathbf{q}_{\mathcal{R}}$.

To increase the comparability through years and across sectoral varying different levels of sectorial disaggregation, we standardise each residual vector by its cross-sectoral sample standard deviation in the corresponding year. This yields dimensionless z-scores. Figure 1 reports kernel-density estimates of the cross-sectoral distributions of these standardised residuals with one panel per residual, with kernel-density per benchmark year.

We identify three features of Figure 1 that are common to all four panels and all nine benchmark years: (i) every distribution is unimodal and approximately symmetric about zero;

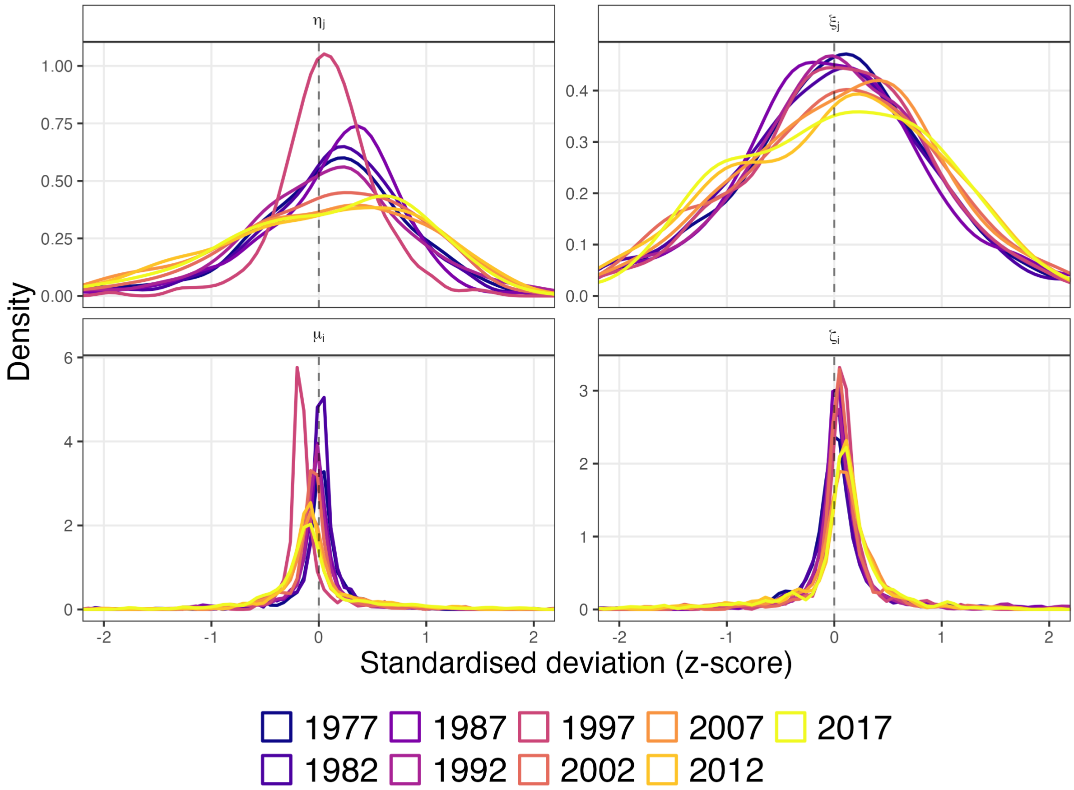

Figure 1: Kernel density estimates of standardised residuals (z-scores) for the four alignment conditions of Corollaries 1–2, U.S. economy, benchmark years 1977–2017. Top row: labour-channel residuals η_j (left) and ξ_j (right). Bottom row: output-channel residuals μ_i (left) and ζ_i (right). Own elaboration with data from BEA (2022, 2025).

(ii) every distribution is centred at or very near zero: no panel exhibits a systematic bias of either sign, so the cross-sectoral mean of each residual is statistically indistinguishable from the value implied by exact alignment; and (ii) the shapes of the densities are stable across the four decades covered, with no visible *secular drift* in either location or scale, i.e., structural changes over the period appear do not appear to manifest themselves via $\boldsymbol{\eta}^T$, $\boldsymbol{\mu}$, $\boldsymbol{\xi}^T$ or $\boldsymbol{\zeta}$.

Nevertheless, the four panels do reveal an steep asymmetry between the labour and output channels. The output-channel residuals $\boldsymbol{\mu}$ and $\boldsymbol{\zeta}$ are concentrated extremely tightly around zero, with modal densities of order 5 and 3 respectively and tails that fall away rapidly. By contrast, the labour-channel residuals $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ are dispersed across a much wider range, with modal densities of order 1 and 0.45 and visibly heavier tails. The same ordering holds within each channel: the *dynamic* residuals ($\boldsymbol{\xi}$, $\boldsymbol{\zeta}$) are modestly more dispersed than the corresponding *geometric* residuals ($\boldsymbol{\eta}$, $\boldsymbol{\mu}$), reflecting the additional amplification introduced by the action of $\mathbf{H}$ in the eigen-equation formulation, but the labour-output asymmetry dominates this within-channel gap by an order of magnitude.

The implication for the channel decomposition of §2.5.1 is clear. Channel (C2) — the output channel, $\mathbf{y} \approx \chi_1 \mathbf{q}_{\mathcal{R}}$ — is tightly satisfied in the U.S. data and contributes a small bound on every subdominant inner product $|\sigma_k|$ through the Cauchy–Schwarz inequality $|\sigma_k| \leq \|\boldsymbol{\alpha}_k\| \, \|\boldsymbol{\chi}_k\|$. Channel (C1) — the labour channel, $\mathbf{v}^T \approx \alpha_1 \mathbf{q}_{\mathcal{L}}$ — is satisfied qualitatively, with a centred and symmetric residual distribution, but appreciably less tightly. The empirical force of Corollary 1 is therefore weaker than that of Corollary 2: the data are more consistent with approximate standard-commodity proportions than with approximate standard-labour proportions.

3.2 The spectrum of A and the rank decay of $\boldsymbol{\alpha}$ and $\boldsymbol{\chi}$

We turn to channel (C3) and its interaction with channels (C1)–(C2). Figure 2 reports the location in the complex plane of the eigenvalues of $\mathbf{A}$ and $\mathbf{H}$, and of the eigenlabour and eigenoutput coordinates $\boldsymbol{\alpha}$ and $\boldsymbol{\chi}$, pooled across all benchmark years and rescaled so that the dominant quantity in each panel lies on the boundary of the unit disk. The dominant Perron–Frobenius eigenvalue is marked in red; all other points are subdominant.

Two regularities are visible. First, the subdominant spectra of both $\mathbf{A}$ and $\mathbf{H}$ are tightly clustered around the origin of the complex plane: the bulk of the subdominant mass lies well inside the disk of radius 0.5, and the modulus of the typical subdominant eigenvalue is small relative to the dominant eigenvalue, so the spectral gap $\lambda_{\mathbf{H},1}/|\lambda_{\mathbf{H},2}|$ is large. This is direct visual evidence for channel (C3). Second, the subdominant eigenlabours and eigenoutputs collapse onto an even tighter neighbourhood of the origin: the clouds in the bottom row of Figure 2 are substantially more concentrated than the eigenvalue clouds in the top row, indicating that the projections of $\mathbf{v}^T$ and $\mathbf{y}$ onto the subdominant eigenspaces are systematically small. Channels (C1) and (C2) operate together with (C3), and reinforce it.

To quantify the rate at which the subdominant coordinates decay, Figure 3 reports the rank-ordered moduli of the four spectral quantities, normalised so that the rank-one entry equals unity, on a logarithmic horizontal axis.

The four panels of Figure 3 differ qualitatively. The eigenvalue panels (top two) display a smooth, approximately concave decay profile, with the second-ranked modulus typically at 0.6–0.8 of the dominant for $\mathbf{A}$ and slightly lower for $\mathbf{H}$, and half-mass reached around rank 10–20. The eigenlabour and eigenoutput panels (bottom two), by contrast, exhibit a sharp cliff between the dominant entry and the subdominant tail: the second-ranked $|\alpha_k|$ already lies between 0.1 and 0.3 of the dominant in every year, and $|\chi_k|$ falls even more steeply — the second-ranked $|\chi_k|$ is typically below 0.5 and frequently below 0.1, and the tail collapses to numerical zero within the first 10–30 ranks. Beyond rank ~ 30, the bottom two panels are visually indistinguishable

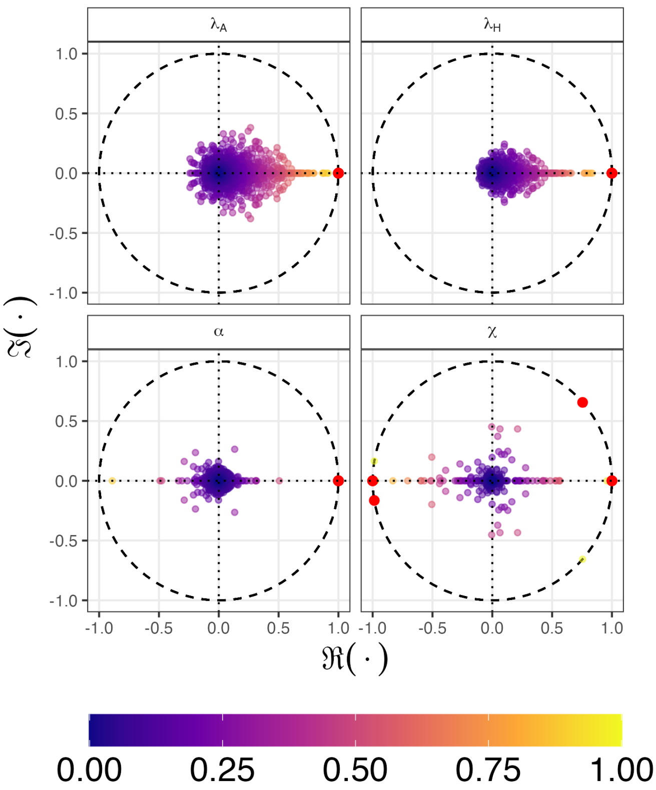

Figure 2: Subdominant spectrum and eigencoordinates in the complex plane (rescaled so that the dominant quantity sits at modulus one), U.S. economy, all benchmark years 1977–2017. Top row: eigenvalues of $\mathbf{A}$ (left) and $\mathbf{H}$ (right). Bottom row: eigenlabours $\boldsymbol{\alpha}$ (left) and eigenoutputs $\boldsymbol{\chi}$ (right). Red markers denote the dominant (P-F) entries; dashed curves are the unit circle. Own elaboration with data from BEA (2022, 2025).

from zero.

The same year-to-year stability seen in Figure 1 re-appears in Figure 3: the curves for the nine benchmark years lie close to one another within each panel, with the partial exception of $\boldsymbol{\chi}$ in 1992 (which decays somewhat more slowly than other years but still exhibits the same cliff structure). The rank decay of the spectral and eigencoordinate quantities is, like the residual distributions, a stable feature of the U.S. technique across the four decades.

Taken together, Figures 1–3 document the three structural regularities announced in the Introduction. Channels (C1)–(C3) all act in the same direction — suppressing the deviation of $w(r)$ from the linear benchmark $1 - r/R$ — and they are not mutually substitutable but complementary: the residual distributions of Figure 1 bound $\|\boldsymbol{\alpha}_k\|$ and $\|\boldsymbol{\chi}_k\|$ (via small geometric and dynamic residuals), while the rank decay and complex-plane concentration of Figures 2–3 bound $|\lambda_{\mathbf{H},k}|$ for the bulk of the subdominant spectrum. The result is that the subdominant inner products σ_k that govern the deviation of $w(r)$ from linearity in (35) are small in two inde-

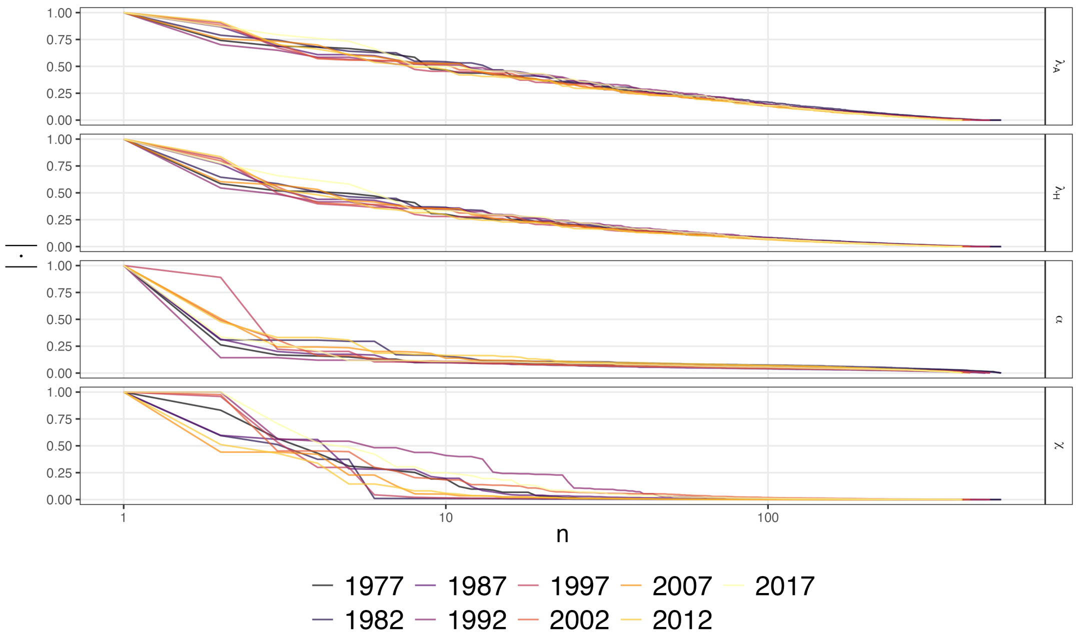

Figure 3: Rank-ordered moduli of the four spectral quantities, normalised so that the dominant entry equals one, U.S. economy, benchmark years 1977–2017. From top to bottom: $|\lambda_{\mathbf{A},k}|$, $|\lambda_{\mathbf{H},k}|$, $|\alpha_k|$, $|\chi_k|$. Horizontal axis is rank k on a logarithmic scale. Own elaboration with data from BEA (2022, 2025).

pendent ways: their factors $\boldsymbol{\alpha}_k$ and $\boldsymbol{\chi}_k$ are small *and* the eigenvalues $\lambda_{\mathbf{H},k}$ that weight them in (33) are small. Near-linearity of the U.S. wage–profit curve is overdetermined by the productive structure.

4 Discussion and conclusion

The wage curve's near-linearity is well-documented across countries and decades: it is the explanation that remains contested. We have argued that the explanation should be sought in the spectral structure of the technique. Proposition 1 characterises exact linearity by a finite set of orthogonality conditions in the eigenbasis of $\mathbf{A}$ — the vanishing of the subdominant inner products $\sigma_k = \boldsymbol{\alpha}_k^T \boldsymbol{\chi}_k$ for $k \geq 2$ — and exposes the standard-labour and standard-commodity conditions of Sraffa as the two corner cases in which the orthogonality is achieved by zeroing one or the other factor of each product. Whenever the σ_k are small but non-zero, the wage curve is near-linear; and (35) makes the relation between the size of the σ_k and the magnitude of the deviation explicit. The three channels (C1)–(C3) of §2.5.1 are the three logically distinct ways the σ_k or their dynamic contribution to $w(r)$ can be made small, and they correspond respectively to the proximity of $\mathbf{v}^T$ to $\mathbf{q}_{\mathcal{L}}$, of $\mathbf{y}$ to $\mathbf{q}_{\mathcal{R}}$, and of the subdominant spectrum of $\mathbf{H}$

to zero.

The empirical analysis of Section 3 establishes that all three channels are active in the U.S. Benchmark Input–Output Accounts across 1977–2017. The four standardised residual distributions in Figure 1 are unimodal, symmetric, and centred at zero in every benchmark year; the complex-plane concentration in Figure 2 and the rank decay in Figure 3 show subdominant eigenvalues clustered near the origin and eigencoordinate moduli collapsing within a small number of ranks. The pattern is stable across four decades, with no visible secular drift. Several substantive points follow.

Overdetermination. The three channels are not redundant ways of saying the same thing: they are independent structural mechanisms, and the U.S. data activate all three simultaneously. This is a stronger empirical finding than the standard-labour or standard-commodity assumption taken alone, since either alone is sufficient for exact linearity but neither alone is observed exactly in the data. Near-linearity in the U.S. technique is overdetermined: it would survive the relaxation of any one channel.

Asymmetry between labour and output. Figure 1 reveals a marked asymmetry: the output residuals are concentrated an order of magnitude more tightly than the labour residuals. The data are quantitatively closer to the standard-commodity case ($\mathfrak{A}_4$, $\mathbf{y} \propto \mathbf{q}_{\mathcal{R}}$) than to the standard-labour case ($\mathfrak{A}_5$, $\mathbf{v} \propto \mathbf{q}_{\mathcal{L}}$), which is non-trivial given that the literature has generally focused on the labour-side regularity. Conceptually, the asymmetry has an economic content worth flagging: the standard-labour condition is a property of the technique alone $(\mathbf{l}, \mathbf{A})$, whereas the standard-commodity condition is a property of the technique *and* the composition of final demand $(\mathbf{y})$. That the latter holds more tightly than the former in the data raises questions about the interaction of demand composition with sectoral input–output structure that lie outside the scope of this paper.

The role of channel (C3). The clustering of the subdominant spectrum of $\mathbf{H}$ near zero is the channel that operates independently of $(\mathbf{l}, \mathbf{y})$. It is a property of the input–output matrix $\mathbf{A}$ alone and reflects the inner sectoral structure of the economy. The empirical content of (C3) is reflected in the upper two panels of Figure 3: the subdominant moduli of $\mathbf{A}$ and $\mathbf{H}$ decay faster than any spectrum-blind null model would predict, and they do so consistently across decades. Why this should be a robust feature of capitalist input–output structures — whether it reflects sparsity, near block-diagonality, or some other structural property of interindustry transactions — is an open question.

Limitations and extensions. Three limitations of the present exercise should be acknowledged. First, the analysis assumes circulating capital only ($\mathfrak{A}_1$–$\mathfrak{A}_3$ exclude fixed capital and joint production); incorporating either generalisation requires moving from $\mathbf{H}$ to a pencil $\lambda\mathbf{B} - \mathbf{A}$ and is beyond this paper. Second, the U.S. data are a single country; whether the overdetermination of channels (C1)–(C3) is a feature of advanced industrial economies in general, or specific to the U.S., requires cross-country replication. Third, the residual distributions of Figure 1 are reported in standardised units; converting the standardised dispersions into quantitative bounds on $|w(r) - (1 - r/R)|$ via (35) is a natural next step. The principal contribution of the paper is to provide a unified spectral framework in which the standard-labour and standard-commodity conditions appear as corner cases of a single characterisation, and the empirical near-linearity of wage curves emerges as a consequence of three independent structural regularities — not one — jointly satisfied by the U.S. technique.

A The Perron–Frobenius Theorem

This appendix collects the statements of the Perron–Frobenius theorem employed throughout the paper. We follow the expositions in Berman and Plemmons (1994), Meyer (2023) and Seneta (1981). Proofs are omitted and may be found in the cited references.

Definition 1 (Non-negative, irreducible, and primitive matrices)**.** *Let* $\mathbf{M} \in \mathbb{R}^{n\times n}$.

(i) $\mathbf{M}$ *is non-negative, written* $\mathbf{M} \geqq \mathbf{O}$*, if* $m_{ij} \geq 0$ *for all* $i, j \in N$.

(ii) $\mathbf{M} \geqq \mathbf{O}$ *is reducible if there exists a permutation matrix* $\mathbf{P}$ *such that*

$$\mathbf{P}^T\mathbf{M}\mathbf{P} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{O} & \mathbf{M}_{22} \end{bmatrix},$$

where $\mathbf{M}_{11}$ *and* $\mathbf{M}_{22}$ *are square submatrices of order at least one.* $\mathbf{M}$ *is irreducible (or indecomposable) if it is not reducible. Equivalently,* $\mathbf{M} \geqq \mathbf{O}$ *is irreducible if and only if* $(\mathbf{I} + \mathbf{M})^{n-1} > \mathbf{O}$.

(iii) An irreducible $\mathbf{M} \geqq \mathbf{O}$ *is primitive if there exists* $k \in \mathbb{N}$ *such that* $\mathbf{M}^k > \mathbf{O}$*. Equivalently,* $\mathbf{M}$ *is primitive if and only if it is irreducible and has a single eigenvalue of maximum modulus.*

Theorem 1 (Perron–Frobenius, primitive case)**.** *Let* $\mathbf{M} \in \mathbb{R}^{n\times n} \geqq \mathbf{O}$ *be primitive. Then:*

(i) There exists a real eigenvalue $\lambda_{\mathbf{M},1} > 0$*, the Perron–Frobenius eigenvalue, such that* $\lambda_{\mathbf{M},1} > |\lambda_{\mathbf{M},i}|$ *for every other eigenvalue* $\lambda_{\mathbf{M},i}$ *of* $\mathbf{M}$*,* $i = 2, \ldots, n$.

(ii) $\lambda_{\mathbf{M},1}$ *is a simple root of the characteristic polynomial of* $\mathbf{M}$.

(iii) *There exist right- and left-hand eigenvectors* $\mathbf{q}_{\mathcal{R}} \in \mathbb{R}^n_{++}$ *and* $\mathbf{q}_{\mathcal{L}} \in \mathbb{R}^{1\times n}_{++}$ *associated with* $\lambda_{\mathbf{M},1}$*, satisfying*

$$\mathbf{M}\mathbf{q}_{\mathcal{R}} = \lambda_{\mathbf{M},1}\mathbf{q}_{\mathcal{R}}, \qquad \mathbf{q}_{\mathcal{L}}\mathbf{M} = \lambda_{\mathbf{M},1}\mathbf{q}_{\mathcal{L}}.$$

These eigenvectors are unique up to positive scalar multiplication. Any non-negative eigenvector of $\mathbf{M}$ *is a positive scalar multiple of* $\mathbf{q}_{\mathcal{R}}$ *(resp.* $\mathbf{q}_{\mathcal{L}}$*), and is associated with* $\lambda_{\mathbf{M},1}$*.*

(iv) *(Collatz–Wielandt formula) The P-F eigenvalue admits the variational characterisation*

$$\lambda_{\mathbf{M},1} = \max_{\mathbf{z}>\mathbf{0}} \min_{i\in N} \frac{(\mathbf{M}\mathbf{z})_i}{z_i} = \min_{\mathbf{z}>\mathbf{0}} \max_{i\in N} \frac{(\mathbf{M}\mathbf{z})_i}{z_i}.$$

(v) *(Power method convergence) Normalising* $\mathbf{q}_{\mathcal{L}}\mathbf{q}_{\mathcal{R}} = 1$*,*

$$\lim_{k\to\infty} \left(\frac{\mathbf{M}}{\lambda_{\mathbf{M},1}}\right)^k = \mathbf{q}_{\mathcal{R}}\mathbf{q}_{\mathcal{L}},$$

and the convergence is geometric at rate $|\lambda_{\mathbf{M},2}|/\lambda_{\mathbf{M},1} < 1$*, where* $\lambda_{\mathbf{M},2}$ *denotes the subdominant eigenvalue.*

Theorem 2 (Perron–Frobenius, irreducible case)**.** *Let* $\mathbf{M} \in \mathbb{R}^{n\times n} \geqq \mathbf{O}$ *be irreducible. Then statements (i)-(iii) of Theorem 1 hold with the following modification of (i): there exists a real eigenvalue* $\lambda_{\mathbf{M},1} > 0$ *such that* $\lambda_{\mathbf{M},1} \geq |\lambda_{\mathbf{M},i}| \forall i = 2,\dots,n$*, with equality possible (the so-called peripheral spectrum). Statement (ii) and the existence of positive left- and right-hand eigenvectors in (iii) remain unchanged. If* $h \geq 1$ *eigenvalues attain the modulus* $\lambda_{\mathbf{M},1}$*, they are the* h*-th roots of unity multiplied by* $\lambda_{\mathbf{M},1}$*; the matrix* $\mathbf{M}$ *is primitive if and only if* $h = 1$*.*

Corollary 3 (Spectral bounds)**.** *Let* $\mathbf{M} \in \mathbb{R}^{n\times n} \geqq \mathbf{O}$ *be irreducible, and let* $\mathbf{M}_{(i)}\mathbf{e} = \sum_{j\in N} m_{ij}$ *and* $\mathbf{e}^T\mathbf{M}^{(j)} = \sum_{i\in N} m_{ij}$ *denote the* i*-th row sum and* j*-th column sum of* $\mathbf{M}$*, respectively. Then:*

(i) *(Frobenius bounds)*

$$\min_{i\in N} \mathbf{M}_{(i)}\mathbf{e} \;\leq\; \lambda_{\mathbf{M},1} \;\leq\; \max_{i\in N} \mathbf{M}_{(i)}\mathbf{e},$$

with analogous bounds for column sums. Equality on either side holds if and only if all row (resp. column) sums are equal.

(ii) *(Monotonicity) If* $\mathbf{N} \geqq \mathbf{O}$ *satisfies* $\mathbf{N} \geqq \mathbf{M}$*, then* $\lambda_{\mathbf{N},1} \geq \lambda_{\mathbf{M},1}$*, with strict inequality whenever* $\mathbf{N} > \mathbf{M}$ *and* $\mathbf{M}$ *is irreducible.*

Corollary 4 (Productive systems and the Leontief inverse)**.** *Let* $\mathbf{A} \in \mathbb{R}^{n\times n} \geqq \mathbf{O}$ *be irreducible. The following statements are equivalent:*

(i) $\lambda_{\mathbf{A},1} < 1$.

(ii) $(\mathbf{I} - \mathbf{A})$ *is a nonsingular* $\mathcal{M}$*-matrix.*

(iii) $(\mathbf{I} - \mathbf{A})^{-1}$ *exists and admits the Neumann expansion* $(\mathbf{I} - \mathbf{A})^{-1} = \sum_{k=0}^{\infty} \mathbf{A}^k > \mathbf{O}$.

(iv) There exists $\mathbf{x} > \mathbf{0}$ *such that* $(\mathbf{I} - \mathbf{A})\mathbf{x} > \mathbf{0}$.

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